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ABSTRACT

This competency-based model has been developed to upgrade the mathematical skills of typical minority students in urban high schools. The central philosophy is that an underprepared student of mathematics can be made computationally literate and mathematically viable in three semesters, whatever the level of preparation. Units covered are Arithmetic and Computation, Algebraic Expressions, Exponents and Radicals, Algebra of Equation Solving, Graphs and Their Interpretation, Geometry and Measurement, Basic Statistics, and Numerical Trigonometry. (MP)

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INTERVENTION MODEL:

MATHEMATICS

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1978

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the Intervention Model Project

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TABLE OF CONTENTS

INTRODUCTION i

ACKNOWLEDGEMENTS iii

ARITHMETIC AND COMPUTATION

Prime Numbers.....Carolyn Anderson 1

Fractions.....Carolyn Anderson 7

Decimals.....Jamye Carter 36

Percents.....James Means 48

Signed Numbers.....Arlys Stritzel 55

INTRODUCTION TO ALGEBRAIC EXPRESSIONS 65
Grady Nelson

EXPONENTS AND RADICALS 77
Thomas Washington, Gwendolyn Wilson

ALGEBRA OF EQUATION SOLVING 93
Jim Factor, Thomas Washington

GRAPHS AND THEIR INTERPRETATION 116
Phillip McNeil

GEOMETRY AND MEASUREMENT 145
Sandra Deloatch

BASIC STATISTICS 181
General Marshall, Alice Simpkins

ELEMENTS OF NUMERICAL TRIGONOMETRY 204
Phillip McNeil, Janie Jordan

Cover Design: Mathematics was a highly developed discipline during the reign of the Pharaohs in Egypt, the most celebrated of the ancient African Empires. Shown here is a segment of hieroglyphic writing much like that which appeared on the interior walls of the Great Pyramids -- enduring testimonials to the Egyptian genius for three-dimensional geometry.

INTRODUCTION

Mathematics, along with communication skills, science, and counseling, is a component in the ISE-sponsored Entry Level Skills Program, known first as the Intervention Model. The academic portion of this program is competency-based. Its purposes are as follows:

(1) Strengthen and refine the Thirteen College Program (TCCP), designed to reduce the attrition rate of students from lower socio-economic circumstances (including ISE's clients in Puerto Rico) and to improve their cognitive competencies;

(2) Equip students with basic level skills necessary to maximize benefits to be derived from upper level courses and produce competitive students capable of acceptable performance on national standardized tests;

(3) Design and implement, along cross-institutional lines, a management information system that will permit the systematic collection, editing and update of data;

(4) Implement, along cross-institutional lines, an evaluation strategy that focuses on the systematic assessment of cognitive growth while sustaining a decade-old emphasis on self-concept improvement and leadership development;

(5) Stimulate well-designed research, including on-campus and cross institutional research.

Draft copies of the teachers' syllabi making up the Entry Level Skills Program were reviewed by representatives from client institutions in April, 1978, at the Deauville Hotel in Miami. The syllabi had been prepared by ISE staff working in conjunction with teacher-researchers from client institutions. Broad guidelines for the syllabi had been provided in a planning document dating from October, 1977, entitled An MIS Approach to the Delivery of Educational Services to Socio-Economically Disadvantaged Students: A Planning Document.

Researchers identified 45 learning objectives within the framework of algebra of equation solving, graphs and their interpretation, selected applications, elements of geometry, elements of numerical trigonometry, and computational statistics. These were brought to the annual curriculum and faculty development conference, most recently held at Dillard University in New Orleans, so that teachers of the quantitative skills from sixteen client institutions could review and refine them for incorporation into a student manual. Further, these teachers were asked to generate appropriate instructional activities and criterion-referenced test items. These activities took place simultaneously with the impacting to administrators and teachers alike of planning, management and evaluation (PME) and research principles,* so that the Entry Level Skills Program would take place under institutional commitment and sound monitoring conditions and be subjected to validation.

Contained in the model are:

(1) A detailed outline of suggested topics for the course sequence laid out in appropriate units

(2) A detailed list of objectives for each unit

(3) Learning packages for each unit designed especially for students

(4) Evaluative items and test-taking hints for each unit

(5) A plan for implementation and management at the individual college.

The central philosophy in the preparation of this model is that an underprepared student of mathematics can be made computationally literate and mathematically viable in three semesters, whatever his level of preparation. Students would be tested for mastery of topics within each unit; and depending on their performance on diagnostic tests, all would probably start somewhere in Unit 1. The diagnostic tests should deal primarily with computation and equation-solving; the unit test items should further stress these same skills within the context of the subject matter in the unit.

*PME and Evaluation manuals form part of a 6-piece educational delivery packet. These manuals not only serve to support program implementation but to introduce PME at the curriculum level and thus ensure proper course management.

ACKNOWLEDGEMENTS

The preparation of this manual represents the contributions of several teachers who attended the Institute for Services to Education 1978 Summer Conference at Dillard University in New Orleans, Louisiana.

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PRIME NUMBERS

Introduction

Prime numbers are important because they are the "building blocks" of all the other numbers. For example, if you were to pick a positive integer -- any positive integer -- it can be written in a unique way as a product of prime numbers. In this section you will learn how to use prime numbers to find the GCD and LCM of any two numbers between 2 and 100. The term "number" in this section refers to natural numbers or positive integers.

Objectives

At the end of this package you should be able to:

(1) List all prime numbers from 2 to 100.

(2) Find all factors of any composite number between 2 and 100.

(3) Write the prime factorization of any number between 2 and 100.

(4) Find the GCD (Greatest Common Divisor) of any pair of numbers between 2 and 100.

(5) Find the LCM (Least Common Multiple) of any pair of numbers between 2 and 100.

Pre-Test

1. List all the prime numbers from 2 to 100.

2. Find all of the prime factors of the following numbers:

   (a) 6 (b) 50

   (c) 22 (d) 72

3. Write the prime factorization of each of the following:

   (a) 8 (c) 24

   (b) 36 (d) 100

4. Find the greatest common divisor (GCD) of each of the following pairs of numbers:

(a) 4, 26 GCD = _____

(b) 8, 23 GCD = _____

(c) 16, 72 GCD = _____

(d) 25, 60 GCD = _____

5. Find the least common multiple (LCM) of each of the following pairs of numbers:

(a) 12, 5 LCM = _____

(b) 12, 4 LCM = _____

(c) 4, 6 LCM = _____

(d) 9, 12 LCM = _____

Key Terms

prime number
composite number
greatest common divisor
factor
prime factorization
least common multiple

Study Material

Let's take a look at the set of natural numbers. With the exception of 1, every natural number can be classified as either a prime or a composite number.

Prime or composite — what exactly do they mean? A prime number is a number that can only be divided exactly by 1 and itself. A composite number is any number greater than 1 that is not prime.

Consider the number 2. Is it prime or composite? Let's look at the numbers that divide 2 exactly. Certainly 1 is such a number, since $2 \times 1 = 2$. Also, 2 is another number, since $1 \times 2 = 2$. These are the only numbers that divide 2 exactly. Hence, 2 is a prime number. Is 3 a prime number? Yes; its only factors are 1 and 3.

Factor? Where did that word come from? A factor is a number that divides another number exactly. In this case 1 and 3 divide 3 exactly and, thus, are factors of 3.

Let's look at 4. Is 4 a prime or a composite number? Looking at its factors we see that 1 is a factor of 4, and 4 is a factor of 4. Is there another factor of 4? 2, of course. Hence, 4 has more than two factors, (namely, three) and must be a composite number.

Name all of the prime numbers less than 12.

Did you decide that 2, 3, 5, 7, and 11 are all the prime numbers? If so, you are correct. Now, for a use of prime numbers:

Consider the numbers 4, 6, 8, and 12. They are all composite numbers, but they also contain prime numbers. For example:

$4 = 2 \times 2$
$6 = 2 \times 3$
$8 = 2 \times 2 \times 2$
$12 = 2 \times 2 \times 3$

In each of the above, we were able to express a composite number as a product of primes. This process is called prime factorization. See if you can write the prime factorization of the following numbers:

24, 39, 49, 54, 62

Answers:

$24 = 2 \times 2 \times 2 \times 3$
$39 = 3 \times 13$
$49 = 7 \times 7$
$54 = 2 \times 3 \times 3 \times 3$
$62 = 2 \times 31$

Practice Problems

a. List all prime numbers less than 100.

b. Write the prime factorization of all composite numbers less than 96.

GCD and LCM

The prime factorization of two numbers can be helpful in using the numbers. For example, consider the numbers 25 and 60. Write their prime factorizations.

$$25 = 5 \times 5$$
$$60 = 2 \times 2 \times 3 \times 5$$

You can see at a glance what factor is common to both 25 and 60. The answer is 5. Notice that 5 is the largest number that is common to both 25 and 60. Now let's look at 20 and 50 and their prime factorizations.

$$20 = 2 \times 2 \times 5$$
$$50 = 2 \times 5 \times 5$$

We see that both 2 and 5 are factors of 20 and 50. Hence, 2 x 5 or 10 is the largest factor of 20 and 50. The largest factor of any two numbers is called the greatest common divisor of the two numbers. The greatest common divisor is often written as the "GCD".

Find the GCD for each of the following pairs of numbers.

14 and 42	68 and 92
45 and 15	12 and 5

Answers:

$14 = 2 \times 7$	$68 = 2 \times 2 \times 17$
$42 = 2 \times 3 \times 7$	$92 = 2 \times 2 \times 23$
GCD = 14	GCD = 4
$45 = 3 \times 3 \times 5$	$12 = 2 \times 2 \times 3$
$15 = 3 \times 5$	$5 = 5 \times 1$
GCD = 15	GCD = 1

Wait a minute! Where did the number "1" come from?

Let's think.

1 is a common factor of any number. Hence, if no other primes are common to the two numbers, their GCD is 1.

There is also another use of the prime factorization of numbers. Let's take another look at 25 and 60.

$$25 = 5 \times 5$$
$$60 = 2 \times 2 \times 3 \times 5$$

Name all the different primes that you see. They are 2, 3, and 5. But 25 contains 5 x 5 and 60 contains 2 x 2. So, let's consider 2 x 2 x 3 x 5 x 5. This equals 300. But 300 contains 25 and it contains 60. Moreover, 300 is the smallest number that contains 25 and 60.

Find the smallest number that contains 20 and 50.

$$20 = 2 \times 2 \times 5$$

$$50 = 2 \times 5 \times 5$$

The primes present are 2 and 5. But we need 2 x 2 and 5 x 5 for 20 and 50 respectively. Hence, 2 x 2 x 5 x 5 = 100 is the smallest number that contains both 20 and 50. The smallest number that contains two given numbers is called the <u>least common multiple</u> (LCM) of the two numbers.

Find the LCM of each of the following pairs of numbers:

14 and 42	68 and 92
45 and 15	12 and 5

Answers:

45 = 3 x 3 x 5	68 = 2 x 2 x 17
15 = 3 x 5	92 = 2 x 2 x 23
LCM = 3 x 3 x 5	LCM = 2 x 2 x 17 x 23
14 = 2 x 7	12 = 2 x 2 x 3
42 = 2 x 3 x 7	5 = 5 x 1
LCM = 2 x 3 x 7	LCM = 2 x 2 x 3 x 5

Practice Problems

a. Find the GCD of each of the following pairs of numbers:

8 and 92	49 and 57
51 and 17	24 and 75

b. Find the LCM of each of the following pairs of numbers:

8 and 12	86 and 100
24 and 51	18 and 63
54 and 98	12 and 95

Post-Test

1. List all prime numbers less than 100.

2. Find all the factors of the following numbers.

   (a) 8 (b) 12

   (c) 86 (d) 100

3. Write the prime factorization of each of the following:

   (a) 24 (b) 18

   (c) 54 (d) 51

   (e) 63 (f) 98

4. Find the greatest common divisor (GCD) of each of the following pairs of numbers:

   (a) 8, 30 GCD = _____

   (b) 10, 40 GCD = _____

   (c) 12, 25 GCD = _____

   (d) 9, 12 GCD = _____

5. Find the least common multiple (LCM) of each of the following pairs of numbers:

   (a) 12, 7 LCM = _____

   (b) 12, 48 LCM = _____

   (c) 12, 15 LCM = _____

   (d) 24, 18 LCM = _____

FRACTIONS

Introduction

The need for fractions in our everyday life is evident. Many situations exist in which objects are actually cut into pieces. For example, a pie may be cut into six equal parts called sixths. A piece of string may be cut into 2 equal parts called halves or into five equal parts called fifths. In this section you will learn to add, subtract, multiply, and divide fractions and mixed numbers.

Objectives

At the end of this section you should be able to do the following:

(1) Name a fraction suggested by a shaded portion of a plane region.

(2) Name a fraction suggested by its position on a line segment.

(3) Identify the numerator and denominator of a fraction.

(4) Write a given whole number as a fraction.

(5) Distinguish between a proper fraction, improper fraction, and a mixed number.

(6) Given a list of 20 proper fractions, identify all sets of equivalent fractions.

(7) Place a list of 5 proper fractions in increasing (decreasing) order.

(8) Find the least common denominator of a set of three proper fractions.

(9) Write any mixed number as an improper fraction and vice versa.

(10) Write the reciprocal for a given fraction or whole number.

(11) Divide a fraction by a natural number.

(12) Divide a whole number by a fraction.

(13) Given any two fractions, find the sum, difference, product, and quotient.

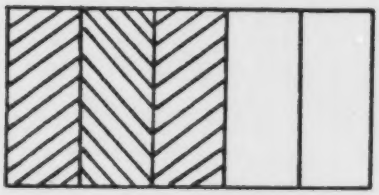
(14) Given any two mixed numbers, find the sum, difference, product and quotient.

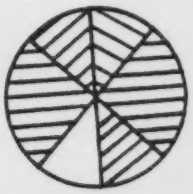
Pre-Test

Directions: Place the answer to each question in the appropriate space.

1. Name the fraction suggested by the shaded portion of each of the following diagrams:

   (a) _____ (b) _____





2. Name the fraction suggested by the arrow on the following line segments:

   (a) _____ (b) _____

3. (a) In the fraction 3/5, the denominator is _____, and the numerator is _____.

   (b) In the fraction 9/10, 9 is the _____, and 10 is the _____.

4. Write the following whole numbers as fractions:

   (a) 3 = _____ (b) 10 = _____

5. Label each of the following as to whether it is a proper fraction, an improper fraction, or a mixed number:

(a) 2 1/5 _____ (b) 41/13 _____

(c) 4/7 _____ (d) 13/41 _____

(e) 13/9 _____ (f) 7 3/8 _____

6. From the following set of 20 fractions, list the sets of equivalent fractions:

1/2, 4/6, 4/12, 18/30, 3/6, 6/10, 10, 15, 2/5, 1/3, 24, 36,

30/50, 8/12, 1/4, 2/3, 2/8, 3/5, 75/150, 2/6, 21/35, 5/10

7. (a) List the following fractions in increasing order:

4/11, 3/11, 10/11, 1/11, 8/11

(b) List the following fractions in decreasing order:

7/10, 7/9, 7/15, 7/8, 7/100

8. Find the least common denominator (LCD) for the following fractions:

(a) 2/3, 1/5, 7/10 LCD = _____

(b) 1/6, 3/10, 4/9 LCD = _____

9. (a) Write the following mixed numbers as improper fractions:

(i) 4 1/3 = _____ (ii) 11 2/5 = _____

(b) Write the following improper fractions as mixed numbers:

(i) 24/7 = _____ (ii) 41/12 = _____

10. Write the reciprocal of each of the following:

(a) 3/8 reciprocal = _____

(b) 12 reciprocal = _____

11. Perform the indicated operations:

(a) 3/10 ÷ 4 = _____

(b) 5 ÷ 3/5 = _____

(c) 1/3 x 2/3 = _____

(d) 6/11 x 3/4 = _____

(e) 3/7 ÷ 5/8 = _____

(f) 6/13 ÷ 2/3 = _____

(g) 2/15 + 7/15 = _____

(h) 5/8 + 3/4 = _____

(i) 11/12 - 7/12 = _____

(j) 6/7 - 2/3 = _____

(k) 2 1/2 x 1 3/4 = _____

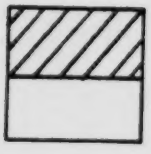
(l) 3 2/5 ÷ 1 3/15 = _____

(m) 4 3/5 + 2 7/10 = _____

(n) 5 3/8 - 2 11/12 = _____

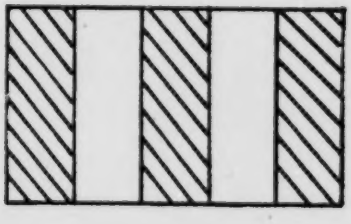
In the circle below, the shaded portion represents 1/4 of the total circle.

In the square, the shaded portion is 1/2 of the total square.



1/3 of this triangle is shaded.

What fraction describes the shaded portion of this rectangle?



If you said 3/5, you are correct.

Key Terms

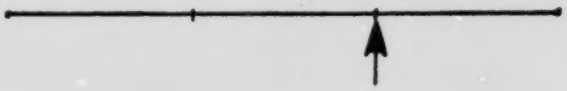
numerator
proper fraction
mixed number
least common denominator
denominator
improper fraction
equivalent fractions
reciprocal

Study Material

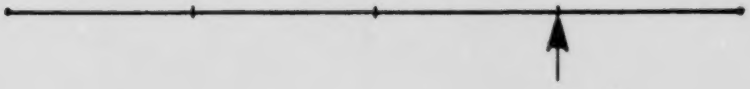
Now let's look at what makes up a fraction. The top number told us how many portions were shaded. This number is called the numerator of a fraction. The bottom number told us how many parts the object had been divided into. This number is called the denominator of the fraction. Since every whole must be divided into something (1,2,3, etc.), then the denominator of a fraction can never be zero.

Now let's take a look at some line segments which have been divided into equal parts. The one below has been divided into two equal parts, so the arrow is pointing to the 1/2 mark on the line segment.

Suppose we divided another line segment into three equal parts. Now the arrow is pointing to the 2/3 mark on the line segment.



Can you determine the fraction that is indicated by the arrow below?

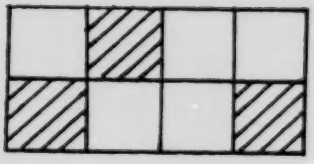


If you said 3/4, you are correct.

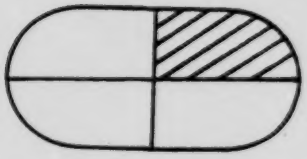
How about trying some practice problems? In each of the following tell what fraction is being represented.

Practice Problems

(a) _____



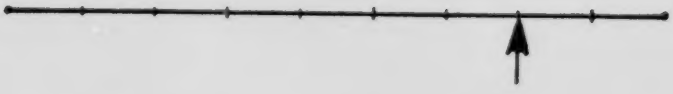
(b) _____



(c) _____

(d) _____

(e) _____



(f) _____

Up to this point, we have considered only fractions which are less than 1. These are called proper fractions. You will notice that their numerators are always less than their denominators.

What about fractions in which the numerator is larger than the denominator? Consider the following picture. What fraction is suggested by the shaded portions?

5/4 is the correct answer. Remember, that the numerator counts the number of shaded parts and the denominator tells us how many parts the whole was divided into.

Consider the following:

The fraction is 2/1. Thus, even whole numbers can be written as fractions. Fractions in which the numerator is greater than the denominator are called improper fractions. Improper fractions are always greater than 1.

Practice Problems

Identify each of the following as an improper or proper fraction.

4/5 _____ 5/6 _____ 7/5 _____ 11/111 _____

The symbol > means "greater than" and the symbol < means "less than". Use these symbols to indicate whether the number to the left in each of the following pairs is greater than or less than 1.

4/9 _____ 1 6/5 _____ 1 7/3 _____ 1

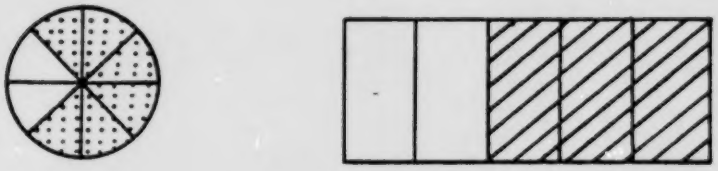
1/3 _____ 1 11/17 _____ 1 119/199 _____ 1

Post-Test

Directions: Place the answer to each question in the appropriate space.

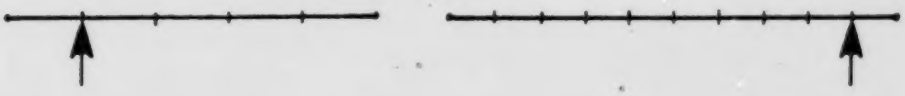
1. Name the fraction suggested by the shaded portion of each of the following diagrams:

(a) (b)



2. Name the fraction suggested by the arrow on the following line segments:

(a) (b)



3. (a) In the fraction 11/10, the numerator is _____, and the denominator is _____.

   (b) In the fraction 6/8, 8 is the _____ and 6 is the _____.

4. Write the following whole numbers as fractions:

   (a) 7 = _____ (b) 1 = _____

5. Label each of the following as to whether it is a proper fraction, an improper fraction, or a mixed number:

(a) 6/9 _____ (b) 3 1/8 _____

(c) 12/9 _____ (d) 1 1/2 _____

(e) 23/17 _____ (f) 17/23 _____

6. From the following set of 20 fractions, list the sets of equivalent fractions:

2/11, 4/8, 44/55, 1/3, 3/4, 3/9, 4/22, 11/22, 13/26, 100/125,

1/9, 80/100, 7/63, 6/8, 4/5, 9/12, 16/88, 75/100, 1/2, 100/900

7. (a) List the following fractions in increasing order:

3/10, 3/4, 3/8, 3/16, 3/7

(b) List the following fractions in decreasing order:

8/17, 1/17, 15/17, 4/17, 9/17

8. Find the least common denominator for the following fractions:

(a) 1/4, 1/5, 1/6 LCD = _____

(b) 5/12, 5/8, 15/36 LCD = _____

9. (a) Write the following mixed numbers as improper fractions:

(i) 1 10/11 = _____ (ii) 12 1/3 = _____

(b) Write the following improper fractions as mixed numbers:

(i) 15/2 = _____ (ii) 56/9 = _____

10. Write the reciprocal of each of the following:

(a) 5 reciprocal = _____

(b) 2/9 reciprocal = _____

11. Perform the indicated operation:

(a) 3/5 ÷ 8 = _____

(b) 2 ÷ 1/2 = _____

(c) 3/8 x 5/8 = _____

(d) 2/15 x 9/10 = _____

(e) 8/9 ÷ 5/9 = _____

(f) 11/12 ÷ 3/4 = _____

(g) 3/8 + 18 = _____

(h) 3/5 + 1/6 = _____

(i) 8/15 - 2/15 = _____

(j) 5/6 - 1/3 = _____

(k) 1 3/4 x 2 3/14 = _____

(l) 2 5/8 ÷ 5 3/4 = _____

(m) 3 1/2 + 10 2/3 = _____

(n) 10 1/4 - 4 5/6 = _____

Mixed Numbers

Not all of the numbers we will encounter will be whole numbers or fractions. Some will be a combination of both. Think about the money that you might have in your wallet. If you have two one dollar bills and four dimes, then you have 2 plus 4/10 dollars. This can be expressed as 2 + 4/10 or 2 4/10. 2 4/10 is called a mixed number.

Where do mixed numbers come from? Let's take another look at improper fractions. 4/3 is a good example. To change 4/3 to a mixed number, divide the denominator into the numerator and write the remainder as a fraction.

Examples:

4/3 = 1 1/3

7/3 = 2 1/3

7/4 = 1 3/4

You can also change a mixed number into an improper fraction. There are three easy steps to follow.

Step 1: Write the mixed number as a whole number plus a fraction.

Step 2: Express the whole number as a fraction with the same denominator as the fractional part of the mixed number.

Step 3: Add the two fractions by adding their numerators and using the same denominator.

Let's try 5 1/3 as an example:

5 1/3 = 5 + 1/3

5 = 15/3

5 1/3 = 5 + 1/3 = 15/3 + 1/3 = 16/3

Practice Problems

a. Change each improper fraction to a mixed number.

11/2 _____ 9/4 _____ 17/3 _____ 5/3 _____

b. Change each of the following mixed numbers to improper fractions.

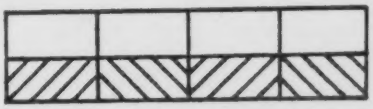
3 1/3 _____ 1 3/4 _____ 5 2/9 _____

Equivalent Fractions

In everyday life you often change a half dollar into two quarters or five dimes. You know that a half dollar, two quarters, and five dimes all have the same value, namely 50¢.

Likewise, you may change one fraction into another fraction having the same value. Such fractions are called equivalent fractions. For example, you may change 1/2 to 4/8 or you may change 4/8 back to 1/2. Thus, 4/8 and 1/2 are two equivalent fractions.

To see this idea in pictures, notice that in the two rectangles below, the shaded area is the same, although in one rectangle it represents 4/8, while in the other it represents 1/2.



A quick way to change 1/2 to 4/8 is to multiply the numerator and denominator of 1/2 by 4, as shown here:

$$1/2 = (1 \times 4) / (2 \times 4) = 4/8$$

Similarly, to change 8/16 to 1/2, you divide both numerator and denominator by 8, as shown here:

$$8/16 = (8 \div 8) / (16 \div 8) = 1/2$$

IF YOU MULTIPLY OR DIVIDE BOTH THE NUMERATOR AND DENOMINATOR OF A FRACTION BY THE SAME NUMBER, YOU GET ANOTHER FRACTION HAVING THE SAME VALUE.

When you divide both numerator and denominator by the same number, you are really reducing the original fraction to one with a smaller number in the numerator and denominator. This is also called reducing to lowest terms. For example, look at 8/16 again. We divided both numerator and denominator by 8 to get an equivalent fraction, 1/2. But the equivalent fraction has a smaller number in both the numerator and the denominator.

How about trying these problems for practice? Be sure to follow the directions carefully.

Practice Problems

a. Find the missing numerator:

$\frac{1}{6} = \frac{}{42}$ $\qquad$ $\frac{4}{5} = \frac{}{15}$

$\frac{1}{4} = \frac{}{20}$ $\qquad$ $\frac{5}{8} = \frac{}{32}$

$\frac{3}{10} = \frac{}{50}$ $\qquad$ $\frac{5}{6} = \frac{}{48}$

b. Reduce these fractions to lowest terms:

8/16 _____ 6/21 _____ 12/18 _____ 18/32 _____

Answers:

a. 7 5 15 12 20 40

b. 1/2 2/7 2/3 9/16

How did you do? If you're ready for some more exercises, try these:

Exercises:

1. In each fraction, tell what number should be put in place of the star so that both fractions will be equivalent.

Examples:

1/2 = */14 (* = 7)

3/11 = */55 (* = 15)

(a) 1/2 = */10

(b) 1/4 = */32

(c) 2/3 = */6

(d) 3/8 = */32

(e) 5/6 = */12

(f) 7/8 = */16

2. Change each fraction to its equivalent with the indicated denominator.

Examples:

3/4 to 20ths — 3/4 = 15/20

7/9 to 54ths — 7/9 = 42/54

(a) 3/4 to 16ths_____

(b) 9/10 to 100ths_____

(c) 1/4 to 60ths_____

(d) 2/9 to 36ths_____

(e) 3/8 to 24ths_____

(f) 1/2 to 12ths_____

(g) 3/5 to 100ths_____

(h) 1/10 to 100ths_____

A FRACTION REDUCED TO LOWEST TERMS IS A FRACTION THAT CANNOT BE WRITTEN WITH A SMALLER NUMERATOR AND DENOMINATOR.

3. Reduce each fraction to lowest terms, and indicate what number you divided into both numerator and denominator.

Examples:

6/8: divide by 2

32/40: divide by 8

5 9/21: divide by 3

(a) 8/36 _____ (b) 40/50 _____ (c) 24/36 _____ (d) 25/75 _____

(e) 15/20 _____ (f) 65/100 _____ (g) 30/36 _____ (h) 6/39 _____

(i) 2 6/8 _____ (j) 5 9/12 _____ (k) 1 10/15 _____ (l) 11 6/32 _____

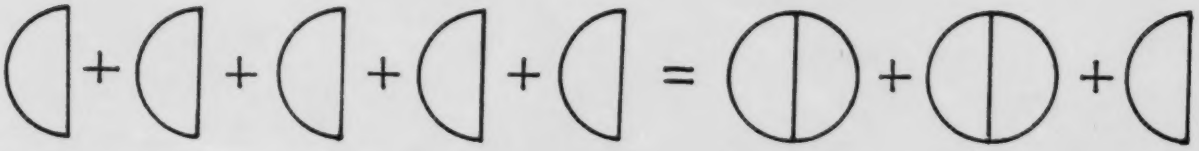
4. In each of the following sets of fractions,circle those which are equivalent.

(a) {3/4, 5/6 9/12, 21/28, 4/13, 2/4, 6/8, 19/38 }

(b) {9/12, 2/6, 4/8, 3/4, 18/24, 45/60, 7/9, 11/13 }

Multiplication of Fractions

Let's consider how we multiply fractions. For example, suppose we wanted to multiply a whole number and a fraction. Consider 5 x 1/2. That is, 1/2 + 1/2 + 1/2 +1/2 +1/2 = 5/2 = 2 1/2. Look at the following picture:



Instead of pictures let's look at 5 x 1/2 and remember that 5 = 5/1. Hence, we have 5 x 1/2 = 5/1 x 1/2 = 5/2 = 2 1/2. Where did 5/2 come from? Remember that multiplication is just repeated addition; 1/2 is added 5 times. But there is an easier way:

TO MULTIPLY FRACTIONS MULTIPLY NUMERATORS TOGETHER AND MULTIPLY DENOMINATORS TOGETHER.

Lets rework our problem using the above rule.

$$5 \times 1/2 = 5/1 \times 1/2 = (5 \times 1)/(1 \times 2) = 5/2 = 2\ 1/2^{**}$$

(**Remember to reduce to lowest terms.)

Suppose you wanted to multiply two fractions like 2/3 x 8/9. Then we have:

$$2/3 \times 8/9 = (2 \times 8)/(3 \times 9) = 16/27$$

Or how about 2/5 x 15/16, which is:

$$2/5 \times 15/16 = (2 \times 15)/(5 \times 16) = 30/80 = 3/8.$$

Of course, not all of our problems will be this easy. So let's try 2/5 x 5/6 x 7/8:

$$\begin{aligned} 2/5 \times 5/6 \times 7/8 &= (2 \times 5)/(5 \times 6) \times 7/8 \\ &= 10/30 \times 7/8 \\ &= (10 \times 7)/(30 \times 8) \\ &= 70/240 = 7/24 \end{aligned}$$

Hope you didn't get lost with all those fancy steps. But let's reverse a minute. Do you remember how we add numbers? We always add two numbers at a time, and the same is true for multiplying numbers; we can only multiply two numbers at one time. Therefore, we have to group the fractions and find the answer for the first two, and then use that answer to multiply the third fraction.

Let's try another type of problem, like 4 1/2 x 5 1/3. These are mixed numbers. But we are supposed to be multiplying fractions, so we must change each mixed number to an improper fraction. Thus, 4 1/2 = 9/2 and 5 1/3 = 16/3. Now let's work this easy problem:

$$4\ 1/2 \times 5\ 1/3 = 9/2 \times 16/3 = (9 \times 16)/(2 \times 3) = 144/6 = 24.$$

Before getting down to business, how about doing the following five problems for review? Be sure to work the problems before looking at the answers and remember to reduce all answers to lowest terms.

Practice Problems

a. 9 x 2/7 = _____

b. 3/8 x 9/10 x 4/9 = _____

c. 10 1/2 x 4 = _____

d. 5/6 x 2/4 = _____

e. 6 2/9 x 4 1/4 = _____

Answers:

a. 2 4/7
b. 3/20
c. 42
d. 5/12

Exercises:

Directions: Multiply and express products in lowest terms.

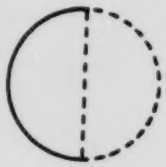
1/3 x 1/4	= _____	12/7 x 49	= _____
1/3 x 3/8	= _____	3/28 x 4/27	= _____
1/5 x 30	= _____	2/3 x 8/3 x 6/5	= _____
3/8 x 4/9 x 7/13	= _____	7 1/2 x 3 3/5	= _____
2 1/3 x 4	= _____	1/2 x 7 2/3	= _____
4 x 2 1/3 x 8/5	= _____	2 1/3 x 3 1/3 x 5 3/7	= _____

What is the total length of lumber needed to make five shelves 3 3/4 feet long?

A pair of shoes is on sale for 1/3 off the regular price. If the list price on the pair of shoes was $33, how much would you save if you bought them?

Division of Fractions

Division is a process of determining how many times one number is contained in another number. For example, the problem "how many 1/4ths are there in 1/2?" can be expressed by writing "1/2 ÷ 1/4 = ". Lets look at a picture first. Here is 1/2 of a pie:



Suppose we had cut the pie in fourths instead of halves Then we would have had this picture:

But since fourths are smaller than halves, our first picture can also be drawn from our second picture. Hence,

and there are two fourths in one half. Thus, 1/2 ÷ 1/4 = 2. We could also draw a picture to find out how many halves there are in three pies. Such as,

for six halves.

That is, 3 ÷ 1/2 = 6.

From this we can see that division is similar to multiplication. Let's try 3/4 ÷ 1/4, or "How many 1/4's are there in 3/4?" This answer is obviously 3.

But how about 5 ÷ 1/5? This would be time-consuming to draw, so let's try another method.

First, look back at our other three examples. The first one was 1/2 ÷ 1/4. Think about how many parts we have if we divide a pie into fourths -- four, and remembering that division is the opposite of multiplication, let's multiply 1/2 by 4 to get 2. Isn't that strange? That was our answer. Will it work for the second example, 3 ÷ 1/2? Is 3 x 2 = 6? Of course it is! Likewise, 3/4 x 4 = 3. Do you think 5 x 5 = 25 is our correct answer for the new problems? It is indeed. Think of five pies, each divided into five slices for a total of twenty-five slices.

Now let's take a look at what we have just done: First we changed 1/5 into 5 and then we multiplied. Changing 1/5 into 5 is the same as inverting 1/5, that is, writing the denominator as the numerator and vice versa.

Consider 2/5 ÷ 2/3. We must first invert our divisor, 2/3 and then multiply. 2/3 inverted is 3/2. Hence 2/5 ÷ 2/3 = 2/5 x 3/2. And we already know how to multiply fractions. Hence, 2/5 ÷ 2/3 = 2/5 x 3/2 = 6/10 = 3/5. Now let's look at 2 1/5 ÷ 3/4. First we change 2 1/5 to an improper fraction, 11/5, then we invert 3/4 to get 4/3, and finally we multiply. So we have 2 1/5 ÷ 3/4 = 11/5; 3/4 = 11/5 x 4/3 = 44/15 = 2 14/15.

How about a rule for division of fractions? A good one, and one that is easy to remember, is IAM: "I" for invert, "A" for and, "M" for multiply. Thus, we have:

I invert
A and
M multiply

How about some practice problems before we do our exercise? Try these five problems before looking at the answers:

Practice Problems

a. 34/45 ÷ 31/36

b. 1/2 ÷ 2/3

c. 9 7/9 ÷ 4 5/6

d. 5/6 ÷ 3/5

e. 6 1/3 ÷ 2 1/2

Answers:

136/155 , 1 7/18 , 3/4 , 2 8/15 , 2 2/87

Exercises:

4 ÷ 2/3	15 ÷ 6/13
6/11 ÷ 7	29/32 ÷ 4
3/5 ÷ 1/4	5/8 ÷ 3/4
16/24 ÷ 11/30	9 1/6 ÷ 10 1/3
10 3/5 ÷ 8 1/2	19/24 ÷ 5/16
6 1/2 ÷ 3 3/4	3 7/8 ÷ 12 1/3

Find the average of the numbers 5/6, 7/15, 8/21.

A board 7 1/4 feet long is to be divided into 8 equal parts. How long will each part be?

Mary bought 17 7/8 yards of material to make skirts for her Brownie troop. If each girl's skirt requires 1 3/4 yards, how many skirts can Mary make?

Fractions:
Addition and Subtraction

Certain problems not present in the multiplication and division of fractions are encountered in the addition and subtraction of fractions -- namely, finding the common denominator.

To see the reason for this, consider the following:

You can add one apple and three apples and get four apples:

$$\begin{array}{r} 1 \text{ apple} \\ +\ 3 \text{ apples} \\ \hline 4 \text{ apples} \end{array}$$

But what about adding one apple and three oranges?

$$\begin{array}{r} 1 \text{ apple} \\ +\ 3 \text{ oranges} \\ \hline ? \end{array}$$

You certainly don't get 4 appleoranges! So, the thing to remember is that in order to add (or subtract) certain things, the things you are adding (or subtracting) must be the same.

The same reasoning carries over to adding and subtracting fractions. You can add one eighth and three eighths and get four eighths ($1/8 + 3/8 = 4/8$), but one eighth and three sevenths are different. You don't add eighths and sevenths to get fifteenths for the same reason you don't add apples and oranges to get apple-oranges! However,

WHENEVER TWO FRACTIONS HAVE THE SAME DENOMINATOR, THEY MAY BE ADDED (OR SUBTRACTED) BY SIMPLY ADDING (OR SUBTRACTING) THE NUMERATORS.

Before going further, try these:

Practice Problems

a. 3/8 + 1/8 _____

b. 1/5 + 2/5 _____

c. 3/11 + 5/11 _____

d. 8/11 - 3/11 _____

e. 5/7 - 2/7 _____

f. 17/35 + 7/35 _____

g. 50/77 + 21/77 _____

h. 32/49 + 18/49 _____

i. 5/6 - 1/6 _____

j. 12/47 + 5/47 + 11/47 + 13/47 _____

To add mixed numbers, you just add the fractions together and the whole numbers together.

Examples:

$$2\ 1/5 + 4\ 2/5 = 6\ 3/5$$

$$12\ 1/8 + 3\ 5/8 = 15\ 6/8 = 15\ 3/4$$

It may be easier to see what to do if the mixed numbers are written in column form:

$$\begin{array}{r} 2\ 1/5 \\ +\ 4\ 2/5 \\ \hline 6\ 3/5 \end{array} \qquad \begin{array}{r} 12\ 1/8 \\ +\ 3\ 5/8 \\ \hline 15\ 6/8 = 15\ 3/4 \end{array}$$

Practice Problems

a. 1 1/3 + 2 1/3 _____

b. 3 12/25 + 4 7/25 _____

c. 5 11/21 - 1 5/21 _____

d. 2 1/10 + 1 3/10 + 4 3/10 _____

e. 1 2/7 + 5 3/7 + 1/7 _____

f. 5 3/4 - 3 1/4 _____

g. Helen's mother used 2 1/4 cups of sugar in a cake and 1 1/4 cups of sugar in the frosting. How many cups of sugar did she use all together for this cake?

h. Joe rode 3 5/12 hours in a train and another 3/12 hour in a bus when he went home for the holidays last year. How many hours did the whole trip take?

Fractions with Different Denominators

Earlier we noted that 1/8 and 3/7 cannot be combined because eighths and sevenths are different. This was true, but what is also true is that 1/8 and 3/7 can be added together if we perform a simple operation on them first. We can change 8ths to 56ths by multiplying both the numerator and denominator by 7. So

$$\frac{1}{8} = \frac{1 \times 7}{8 \times 7} = \frac{7}{56}$$

We can also change 7ths to 56ths by multiplying both the numerator and denominator by 8. For example:

$$\frac{3}{7} = \frac{3 \times 8}{7 \times 8} = \frac{24}{56}$$

What are the advantages of doing this? We couldn't add 1/8 and 3/7 because they had different denominators, but 7/56 and 24/56 have the same denominator and thus can be added together.

$$7/56 + 24/56 = 31/56$$

But since 7/56 = 1/8 and 24/56 = 3/7, this means that

$$1/8 + 3/7 = 31/56.$$

The number 56 is called a common denominator.

Let's consider another example:

The teacher asked Joe and Carolyn to add 1/6 and 3/4. This is the way they did it:

JOE'S WORK

$$\frac{1}{6} = \frac{1 \times 4}{6 \times 4} = \frac{4}{24}$$

$$\frac{3}{4} = \frac{3 \times 6}{4 \times 6} = \frac{18}{24}$$

$$\frac{1}{6} + \frac{3}{4} = \frac{4}{24} + \frac{18}{24} = \frac{22}{24} = \frac{11}{12}$$

CAROLYN'S WORK

$$\frac{1}{6} = \frac{1 \times 2}{6 \times 2} = \frac{2}{12}$$

$$\frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$$

$$\frac{1}{6} + \frac{3}{4} = \frac{2}{12} + \frac{9}{12} = \frac{11}{12}$$

Joe got a common denominator of 24 by multiplying the denominators together. To change 1/6 and 3/4 to 24ths, he multiplied both terms of 1/6 by 4, and both terms of 3/4 by 6. Then, after he performed the addition, he had to reduce the sum 22/24 to 11/12 by dividing both terms by 2.

Carolyn used a smaller common denominator, 12, which also contains both 6 and 4. She got the same answer as Joe, but she did not have to reduce her answer to lowest terms as Joe did.

Both Joe and Carolyn worked the problem correctly; but 12 was better to use because smaller numbers are usually easier to work with. 12 is the smallest, or least common denominator of 1/6 and 3/4, because it is the smallest number that contains both 6 and 4 as factors. It may be wise at this point to refer back to the section on prime factorization to see how to find the smallest number that contains two or more numbers.

Practice Problems

a. 3/8 + 1/4 = _____

b. 7/10 + 1/5 = _____

c. 7/12 - 1/6 = _____

d. 13/20 - 1/2 = _____

e. 1/6 + 5/9 = _____

f. 8/11 - 1/3 = _____

g. 2 1/4 + 5 1/3 = _____

h. 7 4/5 - 2 1/2 = _____

Find the least common denominator for each of the following sets of fractions:

i. 1/3, 5/12, 3/8 _____

j. 2/3, 1/5, 3/10 _____

k. 6/7, 4/49, 3/7 _____

Special Problems Encountered With Adding Mixed Numbers

You may have noticed that up to now we have carefully avoided certain types of problems in adding and subtracting fractions. Well, now is the time to face them and learn how to overcome certain difficulties.

To get an idea of what type problem we are talking about, consider the following examples:

$$\begin{array}{r} 2\ 1/4 \\ +\ 1\ 3/4 \\ \hline 3\ 4/4 \end{array} \qquad \begin{array}{r} 6\ 3/8 \\ +\ 2\ 5/8 \\ \hline 8\ 8/8 \end{array} \qquad \begin{array}{r} 12\ 5/6 \\ +\ 11\ 2/3 \\ \hline \end{array} = \begin{array}{r} 12\ 5/6 \\ 11\ 4/6 \\ \hline 23\ 9/6 \end{array}$$

Notice that in the first two examples the fraction is actually equal to 1. (4/4 = 1 and 8/8 = 1). The third example fraction in the answer is an improper fraction.

What we must do to get the correct answer is to change the fraction to a whole or mixed number and add to the other whole number in the sum.

Therefore, the correct answers to the examples are 4, 9, and 24 1/2. Before doing any practice problems, lets take a look at the next section.

Special Problems Encountered With Subtracting Mixed Numbers

The difficulty in subtracting mixed numbers is related to the fractional part of the mixed numbers. Consider this problem:

$$\begin{array}{r} 5\ 1/4 \\ -\ 2\ 3/4 \\ \hline \end{array}$$

The fraction on the bottom is larger than the fraction on top. Therefore, we can not subtract 3/4 from 1/4. We will have to borrow from the whole number on top. This is the same as when you have to subtract 15 from 42. You can't take 5 from 2 so, you borrow 1 from 4, and rename it 10 to add to the 2 you had, and get 12.

Then it is possible to subtract 5 from 12. In our problem with fractions we are going to do just about the same thing. We can borrow 1 from 5 and then rename the 1 that we borrowed. Since we want to add it to 1/4, it would be best to rename the 1 as 4/4; then we have 5/4 after adding.

$$\begin{array}{r} 5\ 1/4 \\ -\ 2\ 3/4 \\ \hline \end{array} \qquad \begin{array}{r} 4\ 5/4 \\ -\ 2\ 3/4 \\ \hline \end{array} = 2\ 1/2$$

Here is one more example:

$$\begin{array}{r} 5\ 1/5 \\ -\ 1\ 2/7 \end{array} = \begin{array}{r} 5\ 7/35 \\ -\ 1\ 10/35 \end{array} = \begin{array}{r} 4\ 42/35 \\ -\ 1\ 10/35 \\ \hline 3\ 32/35 \end{array}$$

Now here are some problems for you:

Practice Problems

$$\begin{array}{r} 8\ 1/2 \\ -\ 2\ 1/4 \end{array} \qquad \begin{array}{r} 8\ 1/4 \\ -\ 2\ 1/2 \end{array} \qquad \begin{array}{r} 6\ 1/3 \\ -\ 5\ 1/2 \end{array}$$

$$\begin{array}{r} 7 \\ -\ 3\ 1/2 \end{array} \qquad \begin{array}{r} 3\ 1/5 \\ -\ 1\ 7/8 \end{array} \qquad \begin{array}{r} 4\ 1/4 \\ -\ 1\ 1/3 \end{array}$$

$$\begin{array}{r} 4\ 5/9 \\ +\ 4\ 2/3 \end{array} \qquad \begin{array}{r} 1\ 2/11 \\ +\ 1\ 9/11 \end{array} \qquad \begin{array}{r} 8\ 3/5 \\ +\ 5\ 7/10 \end{array}$$

j. $2\ 1/3 + 5\ 1/6 + 4\ 1/2 =$ _____

k. $1\ 1/4 + 3\ 1/2 + 2\ 5/6 + 1\ 2/3 =$ _____

DECIMALS

Introduction

This section introduces you to the concept of decimal fractions. It will enable you to grasp the meaning of decimal fractions, as well as to develop skills in decimal computation.

Objectives

When you have completed this package you should be able to:

(1) Write the correct definition of "decimal fraction".

(2) Given a decimal described verbally, write the corresponding numeral and vice versa.

(3) Given any decimal, write it as a (common) fraction and vice versa.

(4) Place in correct order (increasing or decreasing) any arbitrary list of 5 decimals.

(5) Be able to round off a given decimal to the (a) nearest tenth, (b) nearest hundredth, (c) nearest thousandth, (d) nearest ten thousandth.

(6) Given any two non-negative decimals, find their sum, difference, product, and quotient.

Pre-Test

1. Define a decimal fraction.

2. Write each of the following as a decimal numeral.

   (a) Three hundred twenty-seven and eight tenths
   (b) One hundred eighty-seven thousandths

3. 3.7 is read:

   (a) three and seven tenths
   (b) three and seven hundredths
   (c) thirty-seven hundredths

4. Write the following in words:

   (a) 7.82
   (b) 27.005
   (c) .06

5. Express each of the following decimal fractions as common fractions:

   (a) 0.085
   (b) 3.25
   (c) 0.00025

6. The decimal fraction expression for 5/8 is:

   (a) 0.625
   (b) 0.0625
   (c) 0.00625

7. Arrange the following decimal fractions in increasing order:

   (a) 0.365; 0.037; 0.307, 0.370; 0.360
   (b) 0.5086; 0.586; 0.508; 0.5065; 0.5

8. Round off .15384 to the nearest:

   (a) tenth
   (b) hundredth
   (c) thousandth
   (d) ten thousandth

9. Perform the following operations:

   (a) 3.621 + 83.7

   (b) 28.32 + 7.521

   (c) 5.6 - 3.176

   (d) 354.51 - 38.64

   (e) 23 x 5.4

   (f) 7.05 x 0.006

   (g) 3.62 ÷ 27

   (h) $41\overline{)66.83}$

   (i) 3.62 ÷ 2.7

   (j) $.31\overline{)5.301}$

10. At the candy counter Gertrude weighed all the candy she had in stock. She found that she had 6.25 kilograms of divinity at 1/4 kilogram @50¢; 4.3 kilograms of fudge at 1/4 kilogram @80¢; 1.8 kilograms of chocolate-covered peanuts at ½ kilogram at @90¢; and 5.27 kilograms of mixed nuts at 130¢ a ½ kilogram.

    (a) What was the total weight of the candy and nuts?
    (b) What was the monetary value of the supply of each type of candy?
    (c) What was the monetary value of all the candy?

Key Terms

approximation
common fraction
decimal fraction
decimal point
place value
rational number

Study Material

Using a strip of 1520 metric graph paper, place the numeral 0 at the top left edge. Then count over 10 heavy green lines and place the numeral 1 above the 10th line. Continue this process and place the numeral 2 above the 20th heavy green line. What fractional numbers should the heavy lines between 0 and 1 represent? Between 1 and 2? Label these lines accordingly.

You can answer the above questions by realizing that 10 equal parts make up the segment from 0 to 1; hence, each part must be 1/10 or "one tenth" of a unit. Similarly, there are 10 equal parts in the segment from 1 to 2, each having a length of 1/10.

We can now represent numbers of tenths on our strips from

$$0/10,\ 1/10,\ 2/10,\ \ldots,\ 20/10.$$

A decimal fraction is a symbol for a fractional number named by a fraction whose denominator is a power of 10. Places to the right of the ones place give tenths, hundredths, thousandths, ten thousandths, etc.

A decimal point (period mark) is placed directly to the right of the "ones" place.

The following diagram is quite common and can be used in reading and writing decimal fractions.

10,000	1,000	100	10	1	1/10	1/100	1/1000	1/10,000
ten thousands	thousands	hundreds	tens	ones	tenths	hundredths	thousandths	ten thousandths
10,000	1,000	100	10	1	.1	.01	.001	.0001

Locate .1, .2, .5, 1.2, and 1.8 on your graph strip.

What decimal fractions do the light lines on the graph represent?

Now locate .21, .25, .47, .09, .73, 1.04, 1.79 on the graph strip.

Label all points both as common fraction and as a decimal fractions.

The reading of decimals is patterned after that for fractions in common form. For example, .7 is read "seven tenths". In reading numerals for numbers greater than one, the decimal point is read "and". Thus, 8.78 is read "eight and seventy-eight hundredths".

Now, go back and read aloud each of the decimal fractions you located on the strip of graph paper.

Example:

Decimal	Fraction	Read as:
.2	2/10	two tenths
1.8	1 8/10 or 18/10	one and eight tenths
.47	47/100	forty-seven hundredths
1.004	1 4/1000	one and four thousandt

Note that if the denominator is 10 there is one digit to the right of the decimal point. If the denominator is 100, there are two digits to the right of the decimal point. If the denominator is 1,000, there are three, and so on, as follows:

A decimal fraction such as 0.426 can also be written as:

$$426/1{,}000 = 4(1/10) + 2(1/100) + 6(1/1{,}000)$$

$$\frac{4}{10} + \frac{2}{10^2} + \frac{6}{10^3}$$

Also, it is good practice to write 0.57 instead of .57 since 0 emphasizes the decimal point.

Fractions which have denominators that are factors of ten or factors of a power of ten are also relatively easy to convert to decimal fractions.

For example:

(1) $3/4 = \square/100$

$$\frac{3 \times 25}{4 \times 25} = \frac{\boxed{75}}{100}$$

(2)
$$\begin{array}{r} .75 \\ 4\overline{)3.00} \\ \underline{28} \\ 20 \\ \underline{20} \end{array}$$

In this algorithm (procedure), since the divisor (4) is a whole number, the dividend (3.00) and quotient (.75) have the same number of decimal places. By placing the decimal point of the quotient directly above the decimal point of the dividend, the correct position of the decimal point of the quotient is automatically obtained.

List your decimal fractions on your graph strip from left to right. This gives an increasing ordering of your decimals.

Suppose that you were given these decimal fractions at random. Could you place them in decreasing or increasing order?

Since additional zeros may be added to the right of a decimal fraction without changing it, we can rewrite each of the decimal fractions taking each to equal the largest place value in the given set. Thus, their ordering is more vivid. For example: In the set .1, .2, .5, 1.2, 1.8, .21, .47, 1.04, 1.79 we get .10, .20, .50, 1.20, 1.80, .21, .47, 1.04, 1.79. If we think of .10 as 10, .20 as 20, .50 as 50, etc., our ordering from smallest to largest becomes .1, .2, .21, .47, .5, 1.04, 1.20, 1.79 and 1.8.

Decimal Approximations (Rounding Off)

Is 1.2 closer to 1 or to 2?

Is 1.9 closer to 1 or to 2?

The answers to these questions relate to the process of rounding off a decimal. To round off a decimal to the nearest tenth, for example, we suppress or eliminate all digits in the decimal to the right of the tenths place and:

(1) retain the value of the tenths digit if the hundredths digit is less than 5;

(2) add one to the value of the tenths digit if the hundredths digit is 5 or more.

Example:

(1) 3.47 rounded off to the nearest tenth is 3.5 since in the hundredths place 7 is larger than 5.

(2) 3.42 rounded off to the nearest tenth is 3.4 since in the hundredths place 2 is less than 5.

How would you round off a number to the nearest hundredth? nearest thousandth? etc.

Problem: Round off 2.23075 to the:

(1) nearest tenth:

2.23075 — 3 is less than 5, therefore, 2.23075 to the nearest tenth becomes 2.2

(2) nearest hundredth:

2.23075 - Since 0 is less than 5 it becomes 2.23

(3) nearest thousandth:

2.23075 - since 7 is greater than 5, it becomes 2.231

(4) nearest ten thousandth:

2.23075 - Since 5 is 5 or more, we get 2.2308

Addition and Subtraction

In the addition of whole numbers in a base ten or decimal system, units are added to units, tens to tens, and hundreds to hundreds. It follows that tenths should be added to tenths, hundredths to hundredths, thousandths to thousandths, etc.

The addition and subtraction of decimal fractions can also be related to the addition and subtraction of common fractions.

Example: .3 + .5 = ☐

$$\begin{array}{rr} & .3 = 3/10 \\ + & .5 = 5/10 \\ \hline & 8/10 = .8 \end{array}$$

Keeping the decimal points directly under each other is merely a technique for aligning like digits.

Situations may occur in which the decimals to be added or subtracted differ in regard to the number of places to the right of the decimal point.

Situations such as the following are said to involve "ragged decimals."

3.621 + 3.87 = ☐

5.6 - 3.176 = ☐

4.862 - 2.64 = ☐

Computations to find sums and differences of ragged decimals are made easier by annexing zeros so that all terms will have the same number of places to the right of the decimal and so that addition and subtraction of like units is facilitated.

Therefore, the previously stated problems can be written:

$$\begin{array}{rrr} 3.621 & 5.600 & 4.862 \\ +3.870 & -3.176 & -2.640 \end{array}$$

Now, can you solve them?

Multiplication

Multiplication with decimal fractions can be related to the algorithm for multiplying whole numbers. The only new idea to learn is the location of the decimal point in the product (answer).

- What is the product of .2 and .3? (Hint, rename .2 as 2/10 and .3 as 3/10.)

Using the concepts developed in the multiplication of common fractions, we see that:

.2 x .3 = ☐

2/10 x 3/10 = 6/100

6/100 = .06

Consequently, .2 x .3 = .06

Now try 2.43 x 0.8

Did you rename 2.43 and 0.8 as fractions?

2.43 = 2 43/100 = 243/100

0.8 = 8/10

243/100 x 8/10 = 1944/1,000 = 1.944

Notice that if tenths are multiplied by tenths the result is in hundredths. (1 place x 1 place = 2 places.)

If tenths are multiplied by hundredths the result is in thousandths. (1 place x 2 places = 3 places.)

With similar examples, the generalization can be developed that the number of decimal places in the product is the sum of the decimal places in the factors of the product.

Exercises:

2.49 x 30.6 = _____

67.3 x 1.5 = _____

20 x 2.6 = _____

17.05 = 1.15 = _____

Division

As with the other operations, addition, subtraction, and multiplication for division with decimals is similar to division with whole numbers.

The only difficulty is related to correctly locating the decimal point in the answer or quotient.

(a) When the divisor is a whole number, divide as with whole numbers. The number of decimal places in the quotient is the same as the number of decimal places in the dividend.

Examples:

(1) 8.43 ÷ 3

(2) 56.4 ÷ 8

(3) 258.46 ÷ 14

Solutions:

(1)
$$\begin{array}{r} 2.81 \\ 3\overline{)\,8.43} \\ \underline{6} \\ 24 \\ \underline{24} \\ 3 \\ \underline{3} \\ 0 \end{array}$$

(2)
$$\begin{array}{r} 7.05 \\ 8\overline{)\,56.4} \\ \underline{56} \\ 4 \\ \underline{0} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

(3)
$$\begin{array}{r} 18.46 \\ 14\overline{)\,258.46} \\ \underline{14} \\ 118 \\ \underline{112} \\ 64 \\ \underline{56} \\ 86 \\ \underline{84} \\ 2 \end{array}$$

(b) Suppose, however, that the divisor is not a whole number, such as in the problem 2.82 ÷ .36.

What can you multiply .36 by to make it a whole number?

.36 x☐ = 36

.36 = 36/100

36/100 x☐ = 36

36/100 x☐ = 3600/100 = 36

Since the quotient is unchanged if we multiply both dividend and divisor by 10, we get:

$$\frac{2.82}{.36} \times \frac{100}{100} = \frac{282}{36}$$

Then, performing the division, we get:

$$\begin{array}{r} 7.83 \\ 36\overline{)\,282.00} \\ \underline{252} \\ 300 \\ \underline{288} \\ 120 \\ \underline{108} \\ 12 \end{array}$$

Zeros are added in the dividend as needed depending on the number of decimal places desired for the quotient.

After you fully understand the procedure for establishing an equivalent division expression by multiplying the dividend and the divisor by the same power of 10, the procedure can be shortened by use of the caret ( ). The caret has traditionally been used to indicate the position of the decimal point in the equivalent decimal expression.

For example:

$4.28_{\wedge}\overline{)5.27_{\wedge}6}$ $\qquad$ $23.5_{\wedge}\overline{)6.1_{\wedge}38}$ $\qquad$ $.235_{\wedge}\overline{)6.138_{\wedge}}$

The multiplication of dividend and divisor by the same power of 10 is indicated by the caret, the decimal point is located in the quotient, and the division is performed as with whole numbers.

Exercises:

Perform the following divisions and round off the answers to the nearest hundredth.

3.56 ÷ 2.39 = _____

49.17 ÷ 21.2 = _____

78.1 ÷ 1.53 = _____

18.57 ÷ 5 = _____

Post-Test

1. Why can 1/3 not be represented as a decimal fraction?

2. Sixty-three and forty-eight thousandths written as a decimal fraction is:

   (a) 63.048
   (b) 63.4800
   (c) 63.0048
   (d) .63048
   (e) .6348000

3. Write the following decimal fractions in words.

   (a) 0.8
   (b) .27
   (c) 3.07
   (d) 42.875
   (e) 327.9

4. .15 9/100 written as a decimal fraction is:

   (a) 15.9
   (b) 15.09
   (c) 15.009
   (d) .15900
   (e) .15009

5. Express each of the following decimal fractions as fractions.

   (a) 0.085
   (b) 3.25
   (c) 12.74
   (d) 0.84
   (e) 0.00025

6. The correct ordering from smallest to largest of .06, .62, .625, .613, and .603 is:

   (a) .06, .62, .625, .613, .603

   (b) .06, .603, .613, .62, .625

   (c) .06, .613, .62, .603, .625

   (d) .06, .603, .613, .625, .62

   (e) .06, .603, .62, .613, .625

7. Which of the following distances, in inches, is the greatest?

   (a) 0.8
   (b) 0.65
   (c) 0.009
   (d) 0.019
   (e) 0.300

8. 84.962 rounded to the nearest hundredth is:

   (a) 84.90
   (b) 84.96
   (c) .84
   (d) .85
   (e) 84.97

9. Perform the following operations.

   (a) 28.32 + 7.521

   (b) 0.56 + 0.006

   (c) 0.3 + 5.00311

(d) 2.04 + 4.1

(e) 354.51 - 38.64

(f) 389.27 - 63.99

(g) 691.7 - 8.526

(h) 84.5 - 9.72

(i) 7.05 x 0.006

(j) 0.04 x 6.011

(k) 2.04 x 3.25

(l) 21.7 x 74.65

(m) $41\overline{)66.83}$

(n) 445.64 ÷ 26

(o) $0.42\overline{)4.116}$

(p) 5.301 ÷ 0.31

10. A seamstress purchased 2.40 meters of $1.90 a meter material, 5.40 meters at $3.90 per meter, 7.5 meters at $1 per meter, 25.20 meters of cording at 10¢ per meter, two spools of thread at 70¢ per spool, and 2.2 meters of pleater tape at 60¢ per meter.

    (a) What was the total purchase price for these goods?

    (b) With 5¢ per dollar sales tax, what would her total bill be?

References

Barnes, Bernis; McNeil, Phillip; Richardson, Judith, Interactive Student Inquiry in Mathematics, pp. 65-71. Institute for Services to Education, Washington, D. C., 1974.

PERCENT

Introduction

Knowledge of percent can be useful in many business transactions such as those involving taxes, buying a house or car, borrowing or lending money, installment buying, and sales discounts. Also, the sciences, engineering, and other academic areas make use of percent. This section gives you the opportunity to develop proficiency in the understanding and use of the concept of percent.

Objectives

When you have successfully completed this section, you should be able to:

(1) Give the meaning of the term "percent" and the symbol "%".

(2) Convert any common fraction to a percent and vice versa.

(3) Convert any decimal fraction to a percent and vice versa.

(4) Determine what percent one number is of another.

(5) Find a number which is a given percent of another number.

(6) Find a number such that the percent of that number equals a given number.

(7) Solve word problems using the knowledge of percent.

(8) Solve three basic types of percent problems.

Pre-Test

Note: Answers to all exercises must be calculated to the nearest tenth, if the answer is not an integer.

1. 7 is what percent of 14?

2. 18 is what percent of 6?

3. Change 3¼% to a decimal.

4. Find 9% of 6.

5. Change $\frac{31}{1510}$ to a percent.

6. 2 is what percent of 40?

7. 24 is what percent of 0.12?

8. 249 is 30% of what number?

9. What is 10% of 246?

10. Express 285.6 as a percent.

11. Express 135% as a decimal.

12. Express 0.0075 as a percent.

13. Find 1/3% of $10.

14. Ann and Joe bought a book together. Ann paid 85¢ and Joe paid $2.93. What percent of the book did Ann own?

15. If four cups equal one quart, what percent of a quart are 1 1/2 cups?

16. Shoes marked $37 were sold on sale for $32. What was the percent discount?

17. If the simple interest rate per year is 8 3/4%, how much will Joe have to pay back if he borrows $1,978 for a year?

18. After a 30% discount, a suit sold for $69.50. What was the price before the discount?

19. The cost of a car plus 4% sales tax is $5,279.64. How much was the sales tax?

20. In 1960, a can of milk cost 15¢. In 1978, the can of milk cost 28¢. What is the percent increase from 1960 to 1978?

Key Terms

percent
discount
interest
decimal fraction

Study Material

Computing percents is a useful method of comparing two numbers. To compare the numbers 4 and 8, for example, one forms the fraction 4/8. This fraction is then converted into an equivalent fraction having the denominator 100:

$$\frac{4}{8} = \frac{50}{100}$$

This says that 4 compares to 8 as 50 compares to 100, or "50 per 100". The latter expression is called 50 percent and is written: 50%.

Thus, the term "percent" means "by the hundred" or "per one hundred," and the symbol "%" is used to represent the word "percent".

Practice Exercises on Changing a Fraction to Percent

1. Write each of the following fractions with denominator 100, and then write them as percents: 1/2, 1/3, 1/4, 3/4, 1/20, 2/5, 3/1.

Solutions to Parts of (1):

$1/2 = \frac{50}{100} = 50\%$

$1/3 = \frac{33\ 1/3}{100} = 33\frac{1}{3}\%$

$1/20 = \frac{5}{100} = 5\%$

$3/1 = \frac{309}{100} = 300\%$

Writing a Decimal As a Percent

Remember that a fraction can be written in decimal form by dividing the numerator by the denominator. Now, this decimal may be written in terms of percent by moving the decimal point two places to the right and affixing a % sign.

Examples:

$$1/2 = \frac{50}{100} = .50 = 50\%$$

$$3/4 = \frac{75}{100} = .75 = 75\%$$

$$4/7 = \frac{4(14\ \ 2/7)}{100} = \frac{57\ \ 1/7}{100} = 57\ \ \frac{1}{7}\%$$

Practice Problems

Change 1/4, 2/3, 3/7, 2/5 to percents.

Writing Percent as a Fraction

Recall that percent means per hundred. That is, 25% of a quantity means 25 parts of each hundred parts, or 25/100, which is 1/4 of the quantity.

Example:

Forty percent is 40/100 = 2/5. That is, 40% = 2/5.

Practice Problems

Change 8%, 12 $\frac{1}{2}$%, 37.5%, 75% to fractions.

Solutions to Some Exercises:

$$8\% = \frac{8}{100} = \frac{2}{25}$$

$$37.5\% = \frac{37.5}{100} = \frac{375}{1000} = \frac{3}{8}$$

Writing Percent as a Decimal

Remember that a percent can be written in fractional form with a denominator of 100. Dividing by 100 is equivalent to moving the decimal point two places to the left. So, in order to change percent to a decimal form, move the decimal point two places to the left and drop the percent symbol (%).

Example: Change 25% to decimal form

Solution: 25% = .25

Practice Exercises

Change the following to decimal form: $1\frac{1}{3}\%$, 40%, 125%, 0.12%.

Solutions to Some Exercises:

1 1/3% = $1.3\overline{3}\%$ = $.013\overline{3}$

125% = 1.25

0.012% = .00012

Fill in the missing parts of the following grid:

Percent	Fraction	Decimal
65		
	17/20	
		0.25

Solving Three Types of Percent Problems

Example 1: 2 is 25% of what number?

Solution: Let N represent the unknown number.

Then 25% of N = 2,
.25N = 2

$$N = \frac{2}{.25} = \frac{200}{25} = 8.$$

Example 2: What number is 62.5% of 72?

Solution: Let x represent the unknown number.

Then x = 62.5% of 72

x = .625 x 72 = 45

Example 3: 3 is what percent of 15?

Solution: Let P represent the unknown percent

Then 3 = P of 15

$3 = P \times 15$

$P = \frac{3}{15} = \frac{1}{5} = .20 = 20\%$

Practice Problems

(a) What number is 20% of 30? _____

(b) 30 is what percent of 60? _____

(c) 30 is what percent of 90? _____

(d) What number is 5% of 60? _____

(e) 33 1/3% of 18 is what number? _____

(f) Find $\frac{1}{4}\%$ of 800. _____

(g) 18 is what percent of 24? _____

(h) $7\frac{1}{5}$ is what percent of 8? _____

(i) 1 is what percent of 200? _____

Solutions to Some Exercises:

(d) 3
(e) 6
(f) 2
(h) 90%
(i) $\frac{1}{2}\%$

Post-Test

1. What commission is earned on a $390 sale if the commission rate is 12%?
2. After a 30% discount, an article cost $43.40. Find the original price.
3. Two months ago, the Davis' electric bill was $120. This month the bill was $180. What is the percent increase?
4. The gas tax rate is 40%. What is the tax on $9 worth of gas?
5. If the total cost of a toy including tax is $1.92, and the sales tax is 4%, what is the value of the toy?
6. After a 25% discount, the sale price of a camera is $144. What was its list price?
7. Wayne's income this month is 140% of last month's income. If he earns $650 this month, how much was his income last month?
8. Change the following fractions to percents: 1/4, 1/5, 1/3, 4/7, 5/1, 3/8, 5/8.
9. Change the following percents to fractions: 40%, $33\frac{1}{3}\%$, $133\frac{1}{3}\%$, and 0.2%.
10. Change 66 2/3%, .7%, 235% to decimals.
11. Find 0.5% of 250.
12. 18 is $83\frac{1}{3}\%$ of what number?
13. What number is 20% of 35?
14. 12 1/2 is what percent of 60?
15. Change 0.012 to percent.

References

1. Audio cassettes, Per Cent, by Charles Merrill Company.
2. Any textbook on Junior High School arithmetic.

OPERATIONS ON SIGNED NUMBERS

Introduction

Some concepts in everyday life involve quantities which can be thought of as being on opposite sides of a given reference point. Examples of these are:

(a) temperatures, which may be above 0° Fahrenheit or below 0° Fahrenheit;

(b) altitudes, which may be either above sea level or below sea level;

(c) time, which may be, say, 10 seconds before "blast off" or 10 seconds after "blast off".

(d) financial items, which may constitute a debit or a credit.

Such ideas are usually described by the notion of a positive or negative condition. At the very basic level this suggests a need to study the positive and negative integers and the operations on them. It is toward this end that the following section has been constructed.

Objectives

After you have completed this section you should be able to:

(1) Define the set of integers.

(2) Graph integers on a number line.

(3) Compute the sum of any two integers.

(4) Compute the difference of any two integers.

(5) Compute the product of any two integers.

(6) Compute the quotient of any two integers.

(7) Order subsets of at most five integers.

(8) Identify the absolute value of any integer.

Pre-Test

1. Write in set notation, using brackets, a representation of the set of integers.

2. Draw a graph of the set of integers between -5 and 5 on a number line.

In the following exercises perform the indicated operations and write the answers in the blanks provided.

3. $-7 + 5 =$ _____

4. $-8 + 23 + -10 =$ _____

5. $-9 + -15 =$ _____

6. $-6 - 10 =$ _____

7. $-30/-6 =$ _____

8. $-5 \times -4 =$ _____

9. $-2 \times 15 =$ _____

10. $-115 \div 5 =$ _____

11. $4 \times -6 =$ _____

12. $-(7 + -4) =$ _____

13. Arrange the following from smallest to largest: -2, 10, -4, 0, 7, -1

14. Compute the following absolute values:

    (a) $|-5| =$ _____

    (b) $|7| =$ _____

    (c) $|-(-6)| =$ _____

    (d) $|-8/-4)| =$ _____

15. Indicate the larger of each pair of numbers below by using < or > .

    (a) 7 2 (c) -6 -9 (e) $|-5|$ -3

    (b) -6 4 (d) -8 -1 (f) $|-8|$ 6

Key Terms

natural number
negative integer
absolute value
inverse
opposite
order

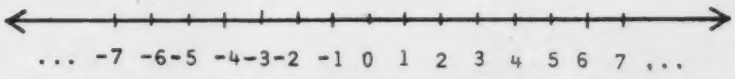
Study Material

The Set of Integers

The set of integers consists of the natural numbers or positive integers $\{1, 2, 3, \ldots\}$, the set of negative integers $\{-1, -2, -3, -4, \ldots\}$, and the set $\{0\}$. Written as a combination or union of these three sets, the integers can be represented by:

$$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

A graph of the integers on a number line appears as follows:



Observe that 0, although it is an integer, is neither positive nor negative. It is the only integer with this property. The integer -5 is called the opposite or additive inverse of 5; and 7 is the opposite of -7. Each integer is the opposite of its negative.

Operations on Integers

In the following chart, we list the rules for adding, subtracting, multiplying and dividing integers. These rules are very important, and you must memorize them in order to correctly perform the operations on the integers.

OPERATIONS CHART FOR INTEGERS

Let a and b represent positive integers.

Addition

a + b	add as usual
a + -b	subtract smaller from larger; use sign of larger
-a + b	subtract smaller from larger; use sign of larger
-a + -b	add the two numbers; use negative sign

Subtraction

a - b	= a + -b, proceed as above
a - -b	= a + b, proceed as above
-a - b	= -a + -b, proceed as above

Multiplication

a x -b	multiply two numbers; use negative sign
-a x b	multiply two numbers; use negative sign
-a x -b	multiply two numbers; use positive sign

Division

-a b	divide b into a; use negative sign
a -b	divide b into a; use negative sign
-a b	divide b into a; use positive sign

Examples

Verify the answers to the following problems using the Operations Chart.

1. $3 + -4 = -1$
2. $-5 + 2 = -3$
3. $-2 + -5 = -7$
4. $6 - 7 = -1$
5. $-3 - 4 = -7$
6. $7 - -5 = 7 + 5 = 12$
7. $3 \times -5 = -15$
8. $-2 \times 9 = -18$
9. $-2 \times -5 = 10$
10. $8/-4 = -2$
11. $-12/2 = -6$
12. $-15/-3 = 5$
13. $-18 \div 2 = -9$
14. $-2 \div -4 = 0.5$
15. $9 \div -2 = -4.5$

Exercises:

Use the chart to write the answers to the following.

(1) $7 + -2 =$ _____

(2) $6 \times -4 =$ _____

(3) $5 - (-2) =$ _____

(4) $8 + -3 =$ _____

(5) $12/-6 =$ _____

(6) $-5 + 4 =$ _____

(7) $-20/-4 =$ _____

(8) $-2 + 3 + -4 =$ _____

(9) $(12/-4) + -2 =$ _____

(10) $7 \quad -14 =$ _____

Answers to Some Exercises

(1) 5, (3) 7, (8) -3, (10) -0.5

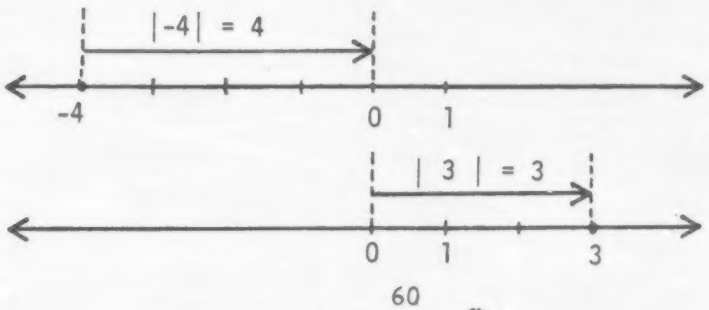
Absolute Value

There are several ways to define the absolute value of an integer, but all the definitions convey the same idea. Two definitions will be given here. The first one relates absolute value to the graphs of integers on the number line.

> IF k IS AN INTEGER, THEN THE ABSOLUTE VALUE OF k IS THE NON-NEGATIVE (POSITIVE OR ZERO) WHOLE NUMBER THAT GIVES THE DISTANCE OF THE GRAPH OF k FROM THE ORIGIN.

The symbol used to denote the absolute value of an integer k is $|k|$. For example, since the graphs of both -2 and +2 are at a distance of 2 units from the origin 0, we can write $|-2| = 2$ and $|2| = 2$; and since the graph of zero is zero units from the origin (it is the origin), then $|0| = 0$.

The Figure below illustrates this first definition of absolute value for the integers -4 and +3.



Our second definition of absolute value does not depend on the number line, but rather on the idea of opposites. Recall that the symbols k and -k represent opposites, so that whether the integer denoted by k is positive, negative, or zero, -k denotes its opposite; similarly, regardless of what the sign of the integer denoted by -k is, k denotes its opposite. Now consider the following definition:

IF k IS AN INTEGER OTHER THAN 0, THEN THE ABSOLUTE VALUE OF k IS THE POSITIVE ONE OF k AND -k.
THE ABSOLUTE VALUE OF 0 IS 0.

Here are some examples to help you interpret this definition:

Example 1: Suppose k = 5. What is $|k|$? The opposite of 5 is -5, and these two numbers form a pair of opposites, one of which is positive and the other negative.

The positive number of the pair is 5; therefore, by our second definition of absolute value, $|5| = 5$.

Example 2: Suppose you want to identify $|-10|$. The opposite of -10 is 10, and the positive number of this pair is 10 ; so $|-10| = 10$.

Exercises:

(1) $|-5| =$ _____

(2) $|6| =$ _____

(3) $|-(-6)| =$ _____

(4) $|8/-4| =$ _____

(5) $|-2 + -1| =$ _____

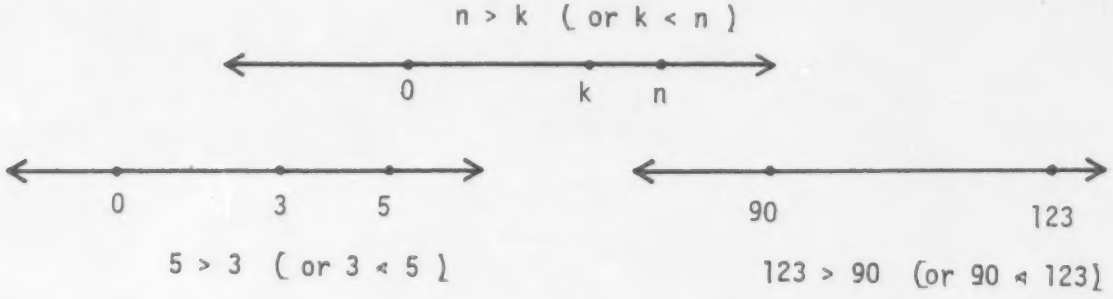
Comparing Integers

You have seen that on the number line, the graphs of the numbers we call integers are the points which are the graphs of the whole numbers 0, 1, 2, 3,... and all the points that are a whole number of unit intervals to the left of the origin, 0. You have also observed that every integer except zero can be paired with a different integer called its opposite, and that the graphs of a pair of opposites are two points the same distance from the origin but on opposite sides of it. Zero is said to be its own opposite.

So far, so good. The integers, however, are not likely to be of much use to you until you know how to compare them. You will study the way in which integers are compared in this section.

By <u>comparing</u> integers, we mean being able to say which of any two different integers is greater than the other. (Clearly, if we can say which integer of a pair is the greater one, we also know which is less than the other.)

You are already familiar with the order of the whole numbers, a proper subset of the set I of integers: recall that if n and k are two whole numbers, then n is greater than k (written $n > k$) if and only if the graph of n lies to the right of the graph of k on the number line. The Figure below shows the picture for whole numbers $n > k$ and for two pairs of specific whole numbers.



Order in the entire set of integers can be defined the same way it is defined above for the whole numbers:

> If <u>a</u> and <u>b</u> are integers, then <u>a</u> > <u>b</u> if and only if the graph of <u>a</u> lies to the right of the graph of <u>b</u> on the number line.

With the ordering of the set I of integers, the trichotomy property of order holds in I just as it does in W, the set of whole numbers; that is:

If a and b are integers, then one and only one of the following statements is true:

$$a < b, \qquad a = b, \qquad \text{or } a > b$$

Exercises:

(1) Use the symbols < or > to show which integer in each pair below is the greater number.

(a) -8,8
(b) 5, -5
(c) -2, 6
(d) 9,0
(e) 3, -7
(f) -14, 5
(g) 0, -1
(h) 1, -2

(2) Arrange the integers in each group below in order from least to greatest.

(a) 4, -8, 6, 2, -5

(b) 1, 2, 0, -3, -1

(3) I am thinking of two integers, a and b. If I make the true statement that a > b, what true statement can you make about -a and -b?

(4) Use the symbols <, >, or = to make each of the sentences below true.

(a) If b > 0 then -b ___ 0.

(b) If b < 0 then -b ___ 0.

(c) If b = 0 then -b ___ 0.

(5) Fill in each box with one of the symbols <, >, or = to make a true sentence.

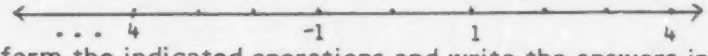
(a) -1 ☐ -10
(b) -10 ☐ 1
(c) 0 ☐ -100
(d) -6 ☐ -6
(e) -(-2) ☐ -2
(f) 0 ☐ -(-1)

Post-Test

1. Write a representation of the set of integers in the braces below:

$\{\quad\quad\}$

2. Indicate the missing integers on the number line below:



3. Perform the indicated operations and write the answers in the blanks provided.

(a) $-6 + 5 =$ _____

(b) $-9 + -7 =$ _____

(c) $12 - -6 =$ _____

(d) $4 - 5 =$ _____

(e) $-8 + -7 + 2 =$ _____

(f) $-3 \times 4 =$ _____

(g) $-4 \times -3 =$ _____

(h) $-60 \div 4 =$ _____

(i) $3 \div -4 =$ _____

(j) $-18/-6 =$ _____

4. Order the following numbers from smallest to largest.

-10, 5, -25, 6, 0, -9

5. Indicate by writing a T or an F which of the following are true statements and which are false statements.

(a) $-7 > -4$ _____

(b) $5 < -6$ _____

(c) $|5| > 9$ _____

(d) $|-3| > 1$ _____

(e) $-15 < -10$ _____

(f) $\frac{1}{2}|-(-6)| < 6$ _____

(g) $-|4| = 4$ _____

(h) $-5 < |-5|$ _____

INTRODUCTION TO ALGEBRAIC EXPRESSIONS

Introduction

This section is designed to lead you from arithmetic into algebra with ease. Algebra is that branch of mathematics which introduces you to the use of letters (alphabets) as numbers. This increases your knowledge in the use of mathematics and introduces you to a higher level of thinking and reasoning. In this section you will gain experience in observing, analyzing, and making decisions (conclusions) while using variables to do abstract addition and multiplication.

Objectives

When you have completed this section, you should be able to:

(1) Given an arbitrary sum of linear expressions in two or more variables x and y, correctly convert the sum to the form ax + by by combining like terms.

(2) Add two fractions with the same single variable denominators and correctly write the answer as a single fraction.

(3) Add two fractions with different single variable denominators and correctly write the answer as a single fraction.

(4) Multiply two first degree binomials to get the correct trinominal expression.

Pre-Test

Write the following by combining all like terms.

1. $5x + 2y + 4x + 7y =$ _____

2. $3x + 4y + 2x - 3y + 5x + y =$ _____

3. $ax + by - cx + dy =$ _____

Write the following sums as a single fraction.

4. $\frac{7}{x} + \frac{8}{x} =$ _____

5. $\frac{a}{8} - \frac{3}{8} =$ _____

6. $\frac{3}{x} - \frac{4}{y} =$ _____

7. $\frac{5}{r} + \frac{7}{st} =$ _____

8. $\frac{25}{p} + \frac{5r}{p} =$ _____

Write the correct trinomial expression for the following products.

9. $(a + 4)(a - 2) =$ _____

10. $(5x + y)(2x - 4y) =$ _____

Key Terms

additive form
variable
algebraic expressions
term
numerical coefficient
literal coefficient
equivalent fractions
numerator
denominator
monomial
binomial
trinomial

Study Material

If you can find the sum and product of 35 and 23, then you are ready for a new adventure. Let us write 35 and 23 in an additive form. That is, 35 as 30 + 5 or 3(10) + 5 and 23 as 20 + 3 or 2(10) + 3. 3(10) + 5 reads "three tens plus 5." Now let us find the sum and product of 35 and 23 in their additive form and compare this result to that obtained in the original form.

Original Form	Additive Form
$\begin{array}{r} 35 \\ +\ 23 \\ \hline 58 \end{array}$	$\begin{array}{r} 3(10) + 5 \\ +\ 2(10) + 3 \\ \hline 5(10) + 8 \end{array}$

Observations:

58 is $5(10) + 8$ in the additive form.

Conclusion:

Both methods are valid for addition.

$$\begin{array}{r} 35 \\ \times\ 23 \\ \hline 105 \\ 70 \\ \hline 805 \end{array} \qquad \begin{array}{r} 3(10) + 5 \\ \times\ 2(10) + 3 \\ \hline 9(10) + 15 \\ 6(100) + 10(10) \\ \hline 6(100) + 19(10) + 15 \end{array}$$

It is not easy to see that these two results are the same. However, $6(100)$ are 600 and $19(10)$ are 190. So $600 + 190 + 15$ is 805. Since $100 = 10^2$, $6(100) + 19(10) + 15$ could be written as $6(10^2) + 19(10) + 15$.

Question:

Do you think the additive form could be used in subtraction and division? (Discuss)

Check Your Skill

Directions: Use the addition form and find the sum and product in the following examples: (Be sure your results are the same).

(1) 21 and 36 _____

(2) 42 and 19 _____

3. 12 and 17 _____

4. Select a pair of numbers and follow the given directions.

5. Show that the results in each multiplication example in the additive form is the same as obtained in the original form.

Coding: Our New Adventure

Let us rewrite 35 and 23 in the additive form in a code. Code: Replace 10 with the letter, t. Simply: $t = 10$. Now $3(10) + 5$ became $3t + 5$ and $2(10) + 3$ became $2t + 3$. The sum and product are as follows:

$$\begin{array}{r} 3t + 5 \\ +\ 2t + 3 \\ \hline 5t + 8 \end{array} \qquad \text{and} \qquad \begin{array}{r} 3t + 5 \\ \times\ 2t + 3 \\ \hline 9t + 15 \\ 6t^2 + 10t \\ \hline 6t^2 + 19t + 15 \end{array}$$

Observation:

The answers are in the additive form.

Conclusion:

1. $5t + 8 = 58$, when $t = 10$.
2. $6t^2 + 19t + 15 = 805$, when $t = 10$.

Check Your Skill

Directions: Write each pair of numbers in the additive form with a code and find the sum and product in each of the following examples. (Don't forget to write down your code.)

1. 27 and 31 _____
2. 49 and 45 _____
3. 16 and 86 _____
4. Select a pair of numbers and follow the given directions.
5. What is the value of your answer in multiplication according to your code?

The idea of using letters and/or other symbols to represent numbers is not new. This has been going on for several generations. Just as we let t represent the number 10, we could have let t represent any number we wished. Since t has this freedom, t is called a variable. Since we can let alphabets represent numbers, we can now add and multiply variables.

$a + a = 2a$ $\qquad$ $a \times a = a^2$

$2a + a = 3a$ $\qquad$ $2a \times a = 2a^2$

$3t + 2t = 5t$ $\qquad$ $3t \times 2t = 6t^2$

$a + b = a + b$ (why?) $\qquad$ $a \times b = a \times b = ab$ (why?)

Remember: $a = 1a = 1a^1$

$a + a = 1a + 1a = 2a$
$a + b = a + b$
$a \times a = 1a \times 1a = 1a^2 = a^2$
$a \times b = ab$

Finding the Sum of Algebraic Expressions

Example 1:

Add: $a + b,\ b + c,\ a + c$

Solution:

$$\begin{array}{r} a + b \\ b + c \\ a \quad + c \\ \hline 2a + 2b + 2c \end{array}$$

Example 2:

Find the sum of: $2a + b - c,\ a - 3b - 5c$ and $4b + 7c$

Solution:

$$\begin{array}{r} 2a + b - c \\ a - 3b - 5c \\ 4b + 7c \\ \hline 3a + 2b + c \end{array}$$

Observations:

1. Like variables are placed under each other.
2. Only like variables are added.

Conclusion:

Like variables can be added to like variables.

Try Your Skill

Find the sum for each of the following:

(1) $4a + c,\ 5b + c,\ a + 6c$

(2) $a - 3b,\ 3a + b,\ 2b - 3c$

(3) $2m + 3n + p,\ 4m - 6n + 2p,\ m + n - p$

(4) Select one example and find the sum.

Find the Product of Two Algebraic Expressions with Two Terms

Example 1:

Multiply: $2a + 3$ and $a + 7$

Solution:

$$\begin{array}{r} 2a + 3 \\ \times\quad a + 7 \\ \hline 14a + 21 \\ 2a^2 + 3a \\ \hline 2a^2 + 17a + 21 \end{array}$$

Example 2:

Multiply: $3x - 2y$ and $2x + 5y$

Solution:

$$\begin{array}{r} 3x - 2y \\ \times\quad 2x + 5y \\ \hline 15xy - 10y^2 \end{array}$$

If you don't understand, then:

1. Reread the examples.
2. Examine each term in each step of the solution.
3. Rework each example.
4. Make yourself a diagram for multiplying.

Important Things to Remember

1. $2a + b$, $m - 3n + p$ or $ax + by$ is called an algebraic expression.
2. $2a$ and b are called terms of the algebraic expression.
3. Algebraic expressions are terms connected by the operation signs: $+, -, \times, \div$.
4. Operation signs tell us what to do.
5. In the term $2a$, 2 is called the numerical coefficient of a.
6. 1 is the _____ _____ of b.
7. In the term ax, a is called the literal coefficient of x or vice versa.
8. b is the literal _____ of y.
9. y is the _____ _____ of b.
10. Numerical coefficients tell us how many or what part. Literal coefficients do not, since they are variables.

Try Your Skill

Directions: Find the product.

(1) $x + 3$ and $x + 10 =$ _____

(2) $a - 9$ and $a + 2 =$ _____

(3) $7n - 1$ and $n - 6 =$ _____

(4) $2x + 3$ and $4x + 7 =$ _____

(5) $2y + 3$ and $2y - 3 =$ _____

Algebraic Expressions

Algebraic expressions are named according to the number of terms the expression contains. Some names are as follows:

One term - Monomial

Two terms -- Binomial

Three terms --- Trinomial

Combining Fractions with a Single Variable Denominator

Like Denominators

Review: Combining fractions with like denominators

Examples:

1. $\frac{1}{5} + \frac{2}{5} = \frac{1+2}{5} = \frac{3}{5}$

2. $\frac{8}{17} - \frac{3}{17} = \frac{8-3}{17} = \frac{5}{17}$

3. $\frac{4}{9} + \frac{4}{9} = \frac{4+4}{9} = \frac{8}{9}$

Observations:

1. The denominators are the same.
2. The numerators are either the same or different.
3. ONLY the numerators are added or subtracted.

Conclusion:

When combining fractions (adding or subtracting fractions) with the same denominator, only the numerators are added or subtracted and the denominator remains the same.

More Examples:

1. $\frac{3}{c} + \frac{5}{c} = \frac{3+5}{c} = \frac{8}{c}$

2. $\frac{-9}{5x} + \frac{7}{5x} = \frac{-9+7}{5x} = \frac{-2}{5x}$

3. $\frac{2x}{y} - \frac{3}{y} = \frac{2x-3}{y}$

4. $\frac{m}{3n} + \frac{p}{3n} = \frac{m+p}{3n}$

Check Your Skill

1. $\frac{2}{a} + \frac{11}{a}$

2. $\frac{7}{3b} + \frac{8}{3b}$

3. $\frac{-6}{m} + \frac{1}{m}$

4. $\frac{23}{5y} - \frac{47}{5y}$

5. $\frac{-3c}{13x} + \frac{4}{13x}$

6. $\frac{7y}{8p} + \frac{5n}{8p}$

Different Denominators

Review: Examples

1. $\frac{1}{3} + \frac{2}{5} = \frac{1 \times 5}{3 \times 5} + \frac{3 \times 2}{3 \times 5} = \frac{5}{15} + \frac{6}{15} = \frac{5+6}{15} = \frac{11}{15}$

2. $\frac{3}{8} + \frac{2}{7} = \frac{3 \times 7}{8 \times 7} + \frac{8 \times 2}{8 \times 7} = \frac{21}{56} + \frac{16}{56} = \frac{21+16}{56} = \frac{37}{56}$

3. $\frac{4}{5} - \frac{4}{9} = \frac{4 \times 9}{5 \times 9} - \frac{5 \times 4}{5 \times 9} = \frac{36}{45} - \frac{20}{45} = \frac{36-20}{45} = \frac{16}{45}$

Fractions can be combined only when the denominators are the same.

Conclusion:

Change fractions with different denominators to fractions with the same denominators, then add or subtract.

(Note: Two fractions having the same value are called equivalent fractions.)

More Examples:

1. $\frac{3}{a} + \frac{4}{b} = \frac{3 \times b}{a \times b} + \frac{4 \times a}{a \times b} = \frac{3b}{ab} + \frac{4a}{ab} = \frac{3b + 4a}{ab}$

2. $\frac{c}{5} + \frac{d}{7} = \frac{7 \times c}{5 \times 7} + \frac{5 \times d}{5 \times 7} = \frac{7c}{35} + \frac{5d}{35} = \frac{7c + 5d}{35}$

3. $\frac{2m}{t} - \frac{n}{y} = \frac{2mxy}{txy} - \frac{nxt}{txy} = \frac{2my - nt}{ty}$

Rule #1: $\frac{a}{b} + \frac{c}{d} = \frac{axd}{bxd} + \frac{bxc}{bxd} = \frac{ad + bc}{bd}$

Rule #2: $\frac{a}{b} - \frac{c}{c} = \frac{axd}{bxd} = \frac{bxc}{bxd} = \frac{ad - bc}{bd}$

Check Your Skill

Directions: Perform the indicated operations:

1. $\frac{x}{3} + \frac{x}{2}$

2. $\frac{1}{x} - \frac{5}{y}$

3. $\frac{4a}{2n} + \frac{b}{4m}$

4. $\frac{4y}{c} - \frac{x}{2d}$

5. $\frac{6m}{3} + \frac{7p}{4m}$

DID YOU USE YOUR RULES?

Post-Test

Directions: Follow the directions given in each example and circle the correct answer.

1. The sum of a + b and 2a + c is:

   (a) 2a + bc (b) 3a + b + c (c) 2a + 2b + c (d) 3a + 3b + 3c

2. The sum of -7m + 2n, 3m + 5n and m - 3n is:

   (a) -3m + 4n (b) -3m + 10n (c) -11m + 4n (d) 11m - 10n

3. $(x + y) + (3x - 4y) + 5y =$

(a) $4x + 9y$ (b) $3x + 10y$ (c) $4x + 2y$ (d) $4x + 10y$

4. $\frac{m}{7} - \frac{4}{7}$ is

(a) $\frac{m - 4}{7}$ (b) $\frac{m - 4}{14}$ (c) $\frac{-4m}{7}$ (d) $m-4$

5. $\frac{3s}{t} + \frac{5p}{t}$ is

(a) $\frac{3s + 5p}{2t}$ (b) $\frac{8sp}{t}$ (c) $\frac{3s + 5p}{t}$ (d) $\frac{8s + p}{t}$

6. The sum of $\frac{pm}{8k}$ and $\frac{jm}{8k}$ is

(a) $\frac{pmj}{16k}$ (b) $\frac{pm + jm}{16k}$ (c) $\frac{pm + j}{8k}$ (d) $\frac{pm + jm}{8k}$

7. $\frac{1}{x} - \frac{2}{y} =$

(a) $\frac{-1}{x-y}$ (b) $\frac{-1}{x+y}$ (c) $\frac{y - 2x}{xy}$ (d) $\frac{2x - y}{xy}$

8. $\frac{5c}{b} + \frac{2}{d} =$

(a) $\frac{7c}{b+d}$ (b) $\frac{7c}{bd}$ (c) $\frac{5cd + 2b}{b+d}$ (d) $\frac{5cd + 2b}{bd}$

9. The sum of $\frac{mn}{s}$ and $\frac{p}{4t}$ is

(a) $\frac{4mnt + sp}{4st}$ (b) $\frac{4mnt + p}{s + 4t}$ (c) $\frac{4mn + sp}{4st}$ (d) $\frac{4mnt + sp}{s + 4t}$

10. The product of $a + 3$ and $a + 5$ is:

(a) $a^2 + 15$ (b) $a^2 + 8a + 15$ (c) $2a^2 + 8$ (d) $2a^2 + 15a + 15$

11. The product of $2t + 3$ and $3t - 4$ is:

(a) $6t^2 - 12$ (b) $6t^2 + 12$ (c) $6t^2 - 17t - 12$ (d) $6t^2 + t - 12$

12. Multiply: $2a - 3b$ and $2a + 3b$

(a) $4a^2 - 9b^2$ (b) $4a^2 - 12ab + b^2$ (c) $4a^2 - 12ab - b^2$ (d) $4a^2 + 9b^2$

EXPONENTS AND RADICALS

Introduction

Exponents are symbols which mathematicians and students use as a kind of shorthand notation. As you will see, a great deal of information can be presented in a small amount of space by using them. The purpose of this section is to introduce exponents through (1) computations with exponents, and (2) the study of exponent laws. In addition, an application of exponents in the area of scientific notation is presented.

Objectives

Upon completion of this section, you should be able to:

(1) Compute the square (2nd power) of any number less than 100.

(2) Compute the cube (3rd power) of any number less than 100.

(3) Find the square root of any perfect square less than 300.

(4) Find the cube root of any perfect cube less than 300.

(5) Approximate the square root of any number less than 300, correct to one decimal place (without a calculator).

(6) Approximate the cube root of any number less than 300, correct to the nearest whole number (without a calculator).

(7) Use the laws of exponents to simplify or evaluate algebraic expressions involving integral exponents.

(8) Write large and small numbers in scientific notation and vice versa.

(9) Use the laws of exponents and scientific notation to perform multiplication and division with decimals.

Pre-Test

1. $7^3 = ?$

   (a) 21
   (b) 7
   (c) 343
   (d) 333

2. $(20)^2 = ?$

(a) 400
(b) 40
(c) 200
(d) 80

3. Select the group which is entirely correct.

(a)	(b)	(c)
$2^3 = 8$	$1^5 = 1$	$9^2 = 81$
$3^4 = 12$	$3^3 = 27$	$2^4 = 16$
$5^2 = 25$	$2^5 = 32$	$1^3 = 3$

4. $\sqrt{169} =$

(a) 12 (b) 13 (c) 40 (d) 43

5. $\sqrt[3]{27} =$

(a) 3 (b) 9 (c) 12 (d) 18

6. Which number below is the best approximation of 90 ?

(a) 7.1 (b) 8.0 (c) 9.5 (d) 9.9

7. Which number below is the best approximation of $\sqrt[3]{200}$?

(a) 3 (b) 5 (c) 7 (d) 9

8. The number 2.68×10^8 equals:

(a) 26,800,000,000 (b) 268,000,000 (c) 26,800,000

9. The number 3,170,000,000,000 equals

(a) 3.17×10^{12} (b) 3.17×10^{10} (c) 3.17×10^{11}

10. Select the group with all three statements correct.

(a)	(b)	(c)
$2x^2y \cdot 3xy^3 = 6x^3y^4$	$4x^4y^3 \cdot 2x^2y = 8x^8y^4$	$3xy^6 \cdot 3x^3y^2 = 9x^4y^8$
$(2a^2b^3)^4 = 16a8_b12$	$(3a^5b^7)^2 = 9a^{10}b^{14}$	$(2a^3b)^3 = 8a^6b^3$
$\left(\frac{8p^2q^4}{4p^4q^3}\right)^2 = \frac{4q^2}{p^4}$	$\left(\frac{9p^5q}{3p^2q^4}\right)^2 = \frac{9p^6}{q^6}$	$\left(\frac{4p^7q^4}{8p^3q^4}\right)^3 = \frac{p^{12}}{8}$

Key Terms

exponent
base
factor
power
radical
scientific notation

Study Material

Numerical Exponents

Exponents are useful as a shorthand notation for strings of repeated multiplication.

Example 1:

$2 \cdot 2 \cdot 2$ is written 2^3.

2 is called the base and 3 is called the exponent in the notation.

2^3 is read "2 to the 3rd power" or "2 cubed".

Example 2:

$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$ is written 3^5.

In this example, 3 is the base and 5 is the exponent.

3^5 is read "3 to the 5th power".

Example 3:

$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$ is written 4^6.

Base = 4
Exponent = 6

4^6 is read "4 to the 6th power".

Example 4:

8^2 stands for $8 \cdot 8$.

Base = 8
Exponent = 2

8^2 is read "8 to the 2nd power" or "8 squared".

In general, for any number, a,

$$\underbrace{a \cdot a \cdot a \ldots a}_{\text{n times}} = a^n$$

a = base
n = exponent

Additional Examples

(a) $(20)^2 = 20 \times 20 = 400$

(b) $(2.3)^3 = (2.3)(2.3)(2.3) = 12.167$

(c) $(1/2)^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

(d) $(-5)^2 = (-5)(-5) = 25$

(e) $(-3)^3 = (-3)(-3)(-3) = -27$

Exercises:

(a) $(13)^2 =$ _____

(b) $(95)^2 =$ _____

(c) $(30)^3 =$ _____

(d) $(4.1)^3 =$ _____

(e) $(1/5)^4 =$ _____

(f) $(5.2)^3 =$ _____

(g) If the exponent is 3 and the base is 7, the number is _____.

(h) If the exponent is 2 and the base is 4, the number is _____.

Radicals

In the formula:

$$3^2 = 9$$

we say that 9 is the 2nd power of 3. In this case, we also call 3 the 2nd root (or square root) of 9. The notation for 2nd root is:

$$3 = \sqrt{9}.$$

The symbol $\sqrt{\ }$ is referred to as a "radical".

Examples:

(a) $4^3 = 64$

Since 64 is the 3rd power of 4, we say that 4 is the 3rd root (or cube root) of 64, written

$$4 = \sqrt[3]{64}$$

(b) $2^4 = 16$

16 is the 4th power of 2, so 2 is the 4th root of 16.

$$2 = \sqrt[4]{16}$$

In general, if

$$a^n = b$$

then, a is the nth root of b, written

$$a = \sqrt[n]{b}$$

Additional Examples:

$\sqrt[3]{8} = 2$ since $2^3 = 8$

$\sqrt{81} = 9$ since $9^2 = 81$

$\sqrt[5]{32} = 2$ since $2^5 = 32$.

Exercises:

(a) $\sqrt{16}$ = _____ (b) $\sqrt{64}$ = _____ (c) $\sqrt{36}$ = _____

(d) $\sqrt[3]{27}$ = _____ (e) $\sqrt[3]{8}$ = _____ (f) $\sqrt[3]{125}$ = _____

Approximating Radicals

Some radicals do not have whole number answers such as those in previous examples.

Problem: What is $\sqrt{70}$? That is, what number has a space equal to 70?

Solution: Since $8^2 = 64$ and $9^2 = 81$, the square of some number between 8 and 9 must equal 70.

$8^2 = 64$

$(?)^2 = 70$

$9^2 = 81$

What is that number? Is it 8.1? 8.2? 8.3?

Checking these possibilities, we find:

$(8.1)^2 = 65.61$

$(8.2)^2 = 67.24$

$(8.3)^2 = 68.89$

$(8.4)^2 = 70.56$

Thus, to the nearest tenth, $\sqrt{70} = 8.4$.

Exercise:

Use this procedure to approximate the following radicals to the nearest tenth.

$\sqrt{2}$ = _____ $\sqrt{6}$ = _____

$\sqrt{3}$ = _____ $\sqrt{7}$ = _____

$\sqrt{5}$ = _____ $\sqrt{8}$ = _____

$\sqrt{20}$ = _____

Answers: 1.4, 1.7, 2.2, 2.4, 2.5, 2.8, 4.4

Problem: What is the $\sqrt[3]{100}$?

Solution: Since $4^3 = 64$ and $5^3 = 125$, the 3rd power of some number between 4 and 5 must equal 100.

$$\begin{aligned} 4^3 &= 64 \\ (?)^3 &= 100 \\ 5^3 &= 125 \\ 4^3 &= 64 \\ (4.1)^3 &= 68.921 \\ (4.5)^3 &= 91.125 \\ (4.6)^3 &= 97.336 \\ (4.7)^3 &= 103.825 \end{aligned}$$

Therefore, 4.6, to the nearest tenth, is the cube root of 100.

Exercise:

Use this procedure to approximate the following radicals to the nearest tenth.

$\sqrt[3]{50} =$ _____ $\sqrt[3]{10} =$ _____ $\sqrt[3]{17} =$

Laws of Exponents

Suppose you were asked to multiply 3^4 times 3^5. Could you write the answer as a power of 3? Solution:

$$3^4 \times 3^5 = (3 \cdot 3 \cdot 3 \cdot 3)\ (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^9,$$

that is,

$$3^4 \times 3^5 = 3^9$$

The preceding example illustrates the use of a fundamental law of exponents:

> **If two powers having the same base are multiplied, the answer is a power having the same base where exponent is the sum of the two original exponents.**

In symbols,

$$a^m \cdot a^n = a^{m+n}$$

We now offer a complete list of the fundamental laws of exponents. You should memorize these laws and use them in the examples and exercises contained in the remainder of this section.

* *

LAWS OF EXPONENTS

1. $a^m \cdot a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. For m a positive integer $(ab)^m = a^m b^m$
5. For m a positive integer and $b \neq 0$,
$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$
6. $a^0 = 1$ for any real number a except 0
7. $a^{-m} = \dfrac{1}{a^m}$ for $(a \neq 0)$

* *

Use the first law of exponents to find answers to the exercises below:

Example:

1. $7x^2y^3z \cdot 4x^4yz^2$

2. $9ab^2c \cdot 6a^2bc^2$

3. $12a^4bc^2 \cdot 7a^4b^2c^4$

4. $15m^2n^3p \cdot 5mnp^2$

Use the second law of exponents to find the answer to the exercises below:

Example:

$$\frac{9a^3b^3c}{3ab^2c^3} = \frac{3a^2b}{c^2}$$

Exercises:

1. $\dfrac{a^4b^5c^6}{a^2b^4c^3}$

2. $\dfrac{20a^6b^7c^8}{5a^3b^3c^4}$

3. $\dfrac{18x^3y^2}{3x^5y^6}$

4. $\dfrac{48x^2}{60x^5}$

Use the third and fourth laws of exponents to find the answers to the exercises below:

Example:

$$(3x^2y^3)^3 = 27x^6y^9$$

Exercises:

1. $(2x^4y^5)^4$

2. $(5m^2n^3p^2)^2$

3. $(6a^3b^4c)^3$

4. $(3x^4y^5c^6)^4$

5. $(a^2)^3$

6. $(3^2)^2$

Use the fifth law of exponents to find the answers to the exercises below:

Example:

$$\left(\frac{2a^2}{3b^3}\right)^3 = \frac{8a^6}{27b^9}$$

Exercises:

1. $\left(\frac{a^2}{b^3}\right)^4$

2. $\left(\frac{2a^3}{5b^2}\right)^2$

3. $\left(\frac{4x^2y^3}{3m^2n^3}\right)^3$

4. $\left(\frac{5a^2b^3c^4}{4x^3y^5z^6}\right)^3$

Use the sixth law of exponents to find the answer to the exercises below:

Example:

$$\frac{3x^0y^0}{6y} = \frac{1}{2}x^0y^0 = \frac{1}{2}$$

Exercises

1. $5xy^0 \cdot 6x^2y^2$

2. $\frac{15ab^2}{3ab^2}$

3. $\left(\frac{8x^3y^5}{9a^4b^9}\right)^0$

4. $\left(\frac{8x^0y^0}{9a^2b}\right)^2$

Use the seventh law of exponents to find the answers to the exercises below:

Write the answers with positive exponents.

Example:

$$\frac{3x^{-2}}{y^{-3}} = 3x^{-2}y^3 = \frac{3y^3}{x^2}$$

Exercises:

1. $(x^{-3}y^{-2}z) \cdot (x^{-1}y^{-2}z^{-3})$

2. $\dfrac{6x^{-6}y^{-2}z^{3}}{3x^{3}y^{-2}z}$

3. $(2x^{-2}y^{-2}z)^{-2}$

4. $(x^{-2}y^{-1})\,(y^{-2}z^{-1})$

Scientific Notation

Often it is necessary to either perform computations with numbers which are very small or very large. This is particularly true in scientific work. Such extreme quantities can be expressed and calculated efficiently by using scientific notation.

A number is said to be expressed in scientific notation when it is written as the product of a number between 1 and 10, and a power of 10. Simply move the decimal point to the right or left of the chosen number, obtaining a number between 1 and 10. Multiply this number by 10 with an exponent numerically equal to the number of places the decimal point has been moved. The exponent is positive if the decimal point has been moved to the left, and negative if the decimal point has been moved to the right.

Examples:

1. Write 287 in scientific notation.

Solution: Move the decimal point two places to the left in 287. You should get 2.87. This amounts to dividing by $100 = 10^2$. To offset this, multiply 2.87 by 10^2. This gives 2.87×10^2, which is the answer.

2. Write 0.0287 in scientific notation.

Solution: Move the decimal point two places to the right in 0.0287. You should get 2.87. This amounts to multiplying by $100 = 10^2$. To offset this, divide by 10^2 or multiply by 10^{-2}. This gives 2.87×10^{-2}, which is the answer.

Exercises:

Write the following numbers in scientific notation; that is, a number between 1 and 10 multiplied by 10 to a whole power.

1. 3620
2. 0.123
3. 8,370,000
4. 0.00073
5. 0.0000000068
6. 687,000,000,000

Answers

1. 3.62×10^{3}
2. 1.23×10^{-1}
3. 8.37×10^{6}
4. 7.3×10^{-4}
5. 6.8×10^{-9}
6. 6.87×10^{11}

It is equally desirable that you learn to change from scientific notation to ordinary form. Suppose you are asked to write 5.63×10^{4} in ordinary form. What should you do? You could write 10^{4} as 10,000 and multiply it by 5.63. You wouldn't want to take that approach all the time, however, because there is an easier way. Look at the positive exponent 4. It tells you that the decimal point was moved 4 places to the left when the answer was written in scientific notation. Hence, you want to move the decimal point 4 places to the right, getting

$$5.63 \times 10^{4} = 56{,}300 .$$

Another example:

Write 3.22×10^{-4} in ordinary form.

The negative exponent 4 tells you that the decimal point was moved four places to the right when the number was written in scientific notation. So you want to move your decimal point 4 places to the left, obtaining

$$3.22 \times 10^{-4} = 0.000322$$

Exercises:

Write the following numbers in ordinary form.

1. 3.27×10^{6}
2. 8.32×10^{-4}
3. 7×10^{-5}
4. 9.768×10^{4}
5. 4.6×10^{-14}
6. 6.163×10^{22}

Answers:

1. 3,270,000
2. 0.000832
3. 0.00007
4. 97680
5. 0.000000000000046
6. 61,630,000,000,000,000,000,000

Finally, if numbers are expressed in scientific notation, it is possible to multiply and divide them quite conveniently by using the laws of exponents.

Examples:

1. $0.00064 \times 8000 = 6.4 \times 10^{-4} \times 8 \times 10^{3}$
   $= 6.4 \times 8 \times 10^{-4} \times 10^{3}$
   $= 51.2 \times 10^{-1}$
   $= 5.12$
2. $76{,}000 \div 0.0004 = 7.6 \times 10^{4} \div 4 \times 10^{-4}$

2. 76,000 0.0004 $= 7.6 \times 10^4 \quad 4 \times 10^{-4}$

$$= \frac{7.6 \times 10^4}{4 \times 10^{-4}}$$

$$= \frac{7.6}{4} \times \frac{10^4}{10^{-4}}$$

$$= 1.9 \times 10^{4-(-4)}$$

$$= 1.9 \times 10^8$$

$$= 190,000,000$$

In these examples you will notice that we have utilized some of the laws of exponents as well as what you learned about scientific notation.

Exercises:

Use scientific notation to perform the following computations:

1. 0.0004 x 120,000 x 200

2. $\frac{54.000}{0.000006}$

3. $\frac{8000 \times 0.004}{1600}$

Answers:

1. 9600

2. 9,000,000,000

3. 0.02

Post-Test

1. $7^2 =$ _____

   (a) 14
   (b) 21
   (c) 49
   (d) 56

2. $(30)^3 =$

   (a) 270
   (b) 900
   (c) 9,000
   (d) 27,000

Select the group which is entirely correct:

3.

(a)	(b)	(c)
$3^3 = 27$	$3^4 = 81$	$1^7 = 7$
$2^4 = 8$	$6^3 = 216$	$2^6 = 64$
$1^6 = 1$	$5^3 = 125$	$3^5 = 243$

4. $\sqrt{121} =$

   (a) 11 (b) 12 (c) 20 (d) 21

5. $\sqrt[3]{216} =$

   (a) 7 (b) 6 (c) 5 (d) 4

6. Which of the following is the best approximation of $\sqrt{110}$?

   (a) 9.0 (b) 9.5 (c) 10.0 (d) 10.5

7. Which of the following is the best approximation of $\sqrt[3]{126}$?

   (a) 3 (b) 5 (c) 7 (d) 15

8. The number 3.52×10^8 equals:

   (a) 352,000,000 (b) 35,200,000 (c) 3,520,000

9. The number 672,000,000,000 equals:

   (a) 6.72×10^9 (b) 6.72×10^{11} (c) 6.72×10^{10}

10. Select the group with all three statements correct.

(a)

$3x^2y \cdot 4xy^3 = 12x^3y^3$

$(2a^3b^2)^3 = 6a^6b^9$

$\left(\frac{4a^2b^3}{2a^4b^2}\right)^2 = \frac{4b}{a^4}$

(b)

$6a^3b^2 \cdot 4a^2b^3 = 24a^5b^5$

$(3x^4b^3)^3 = 27x^{12}b^9$

$\left(\frac{12x^2y^5}{4x^3y^6}\right)^2 = \frac{9}{x^2y^2}$

(c)

$5x^4y^7 \cdot 8x^3y = 40x^7y^8$

$(4a^3b)^3 = 64a^6b$

$\left(\frac{15x^4y^5}{20x^3y^2}\right)^2 = \frac{9x^2y^9}{4}$

ALGEBRA OF EQUATION SOLVING - PART I

Introduction

This section will help you to develop the step-by-step skills necessary to solve first and second degree equations in one variable. It will illustrate the use of equations in solving word problems and proportions.

Objectives

When you have completed this section you should be able to do the following:

(1) Solve any first degree equation in one variable and check the answer.

(2) Convert any verbal first degree equation into the correct symbolic equation and solve the equation.

(3) Translate a word problem into the correct symbolic equation and solve.

(4) Express any fraction as a ratio and vice versa.

(5) Determine whether two given ratios form a proportion.

(6) Given three numbers of a proportion, find the fourth number.

(7) Convert any verbal second degree equation in one variable into the correct symbolic equation.

(8) Write from memory the Quadratic Formula.

(9) Solve any quadratic equation (with real roots) by using the Quadratic Formula.

Pre-Test

1. Solve for x: $-2/3x = 6$

2. Solve for x: $3x + 2 = 3$

3. Solve for y: $6y + 3 = 4y + 2$

4. Solve for x: $6(x + 2) = 36$

5. Solve for n: $6/n = 3/7$

6. If a board 36 feet long is cut into two parts and the larger part is three more than twice the smaller, what is the length of each part?

7. Do 3:6 and 9:18 form a proportion? Why or why not?

8. Solve for x by using the quadratic formula: $2x^2 + 6x - 1 = 0$

Write the following English words into an equation and solve:

9. Six times x is four less than three times x.

10. Two more than three times x plus x^2 is equal to zero.

Key Terms

variable
equation
solve
ratio
proportion
extremes
means
Quadratic equation
Quadratic formula

Study Material

Equation Solving

Draw a square. Subdivide this square into 4 equal squares (Figure 1). Now draw a square identical to the first square, subdivide it into 4 equal squares, and draw in its diagonals (Figure 2).

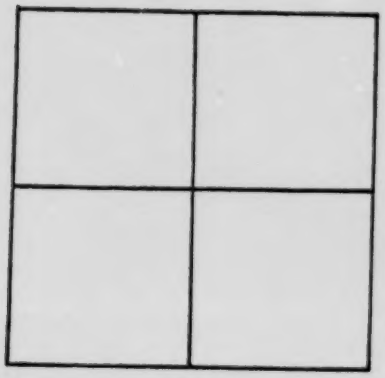


Figure 1

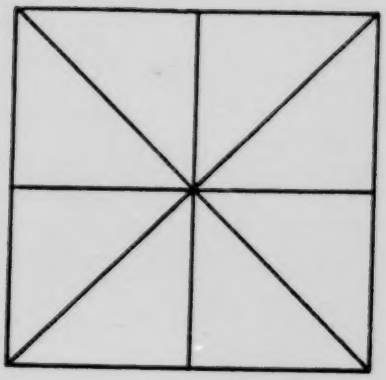


Figure 2

What is the relationship between the four squares of Figure 1 and the eight squares of Figure 2?

Answer: They both cover the same or equal areas.

If S is allowed to stand for one of the 4 squares in Figure 1, how can this figure be represented?

Answer: 4S

If T is allowed to stand for one of the triangles in Figure 2, how can this figure be represented?

Answer: 8T

Since both figures started with identical squares, we have $4S = 8T$.

Now suppose that a triangle in Figure 2 is one square unit. We then have 4S equal to 8 square units. That is, $4S = 8$. This last expression gives us a first degree equation in one variable (the variable is S here). It algebraically poses this question: Since 4S is equal to 8 square units, how many square units are one of the 4 squares equal to? The answer to the question can be easily obtained by just looking at Figure 1 and Figure 2 . The answer is: 2 square units. But many times we may be left with only the equation to solve and not have the benefit of the figures. This is the situation we must learn to deal with.

1.14

Recall the concept of reciprocity. Note that the reciprocal of 3 is 1/3; the reciprocal of -2/3 is -3/2; the reciprocal of 1/6 is 6/1 or 6. In general, the reciprocal of a number is obtained by interchanging the numerator and denominator of that number. If any number is multiplied by its reciprocal, the result is 1. In Figure 2, if one of the eight triangles (i.e., 1/8 of the original square) is laid down next to itself 8 times, the result will again cover an area equal to the area of the original square. That is, $1/8 \times 8 = 1$. Similarly, if two of the eight triangles (i.e., 2/8 of the original square) are laid down next to themselves 8/2, or 4, times the result will again cover an area equal to the area of the original square. That is, $2/8 \times 8/2 = 1$.

Using these facts, let us solve the above equation using this property of reciprocals. Since the reciprocal of 4 is 1/4, we therefore have:

$$1/4 \cdot 4S = 1/4 \cdot 8$$

$$1 \cdot S = 8/4$$

$$S = 2$$

Note 1: Since the left-hand side of the equation is multiplied by 1/4, the right-hand side must also be multiplied by 1/4 in order to maintain the equality.

> **In general, whatever is done to one side of the equation must also be done to the other side.**

Note 2: $1 \cdot S = S$, that is $1 \cdot S$ is the same as S.

Exercise 1: Using reciprocals, solve the following problems:

(a) $5x = 10$

(b) $\frac{-x}{3} = 2$

(c) $\frac{4}{5x} = 8$

(d) $\frac{3x}{4} = 7$

Solutions:

(a) The reciprocal of 5 is 1/5, therefore

$$1/5 \cdot 5x = 1/5 \cdot 10$$
$$x = 2$$

(b) Note $-x/3$ is the same as $-1/3\ x$. Since the reciprocal of $-1/3$ is $-3/1$ or -3, we have

$$-3 \cdot (-x/3) = -3 \cdot 2$$
$$x = -6$$

(c) The reciprocal of 4/5 is 5/4, therefore

$$5/4 \cdot 4/5\ x = 5/4 \cdot 8$$
$$x = 10$$

(d) Note $3x/4$ is the same as $3/4\ x$. Since the reciprocal of 3/4 is 4/3, we have

$$4/3 \cdot 3x/4 = 4/3 \cdot 7$$
$$x = 28/3$$

To solve an equation means to get the variable on a side by itself.

The Use of Opposites in Equation Solving

Equations can be solved by using the opposite (i.e., additive inverse) of a number. The opposite of -2 is 2. The opposite of 1/6 is -1/6. The opposite of 3.9 is -3.9.

Consider: $x + 6 = 9$

To solve this equation, add the opposite of 6 to both sides; that is, add -6 to both sides. Therefore,

$$x + 6 - 6 = 9 - 6$$
$$x = 3$$

Consider: $y - 7.2 = 3.3$

In order to solve this equation, add the opposite of -7.2 to both sides; that is add 7.2 to both sides. Therefore

$$y - 7.2 + 7.2 = 3.3 + 7.2$$
$$x = 10.5$$

Exercise 2: Using opposites, solve the following equations:

(a) $y - 7 = 9$ (b) $x + 3.1 = 8.4$ (c) $z + 1/3 = 5$

Solutions:

(a) The opposite of -7 is 7, therefore

$$y - 7 + 7 = 9 + 7$$
$$y = 16$$

(b) The opposite of 3.1 is -3.1, therefore

$$x + 3.1 - 3.1 = 3.4 - 3.1$$
$$x = 5.3$$

(c) The opposite of 1/3 is -1/3, therefore

$$z + 1/3 = 5$$
$$z + 1/3 - 1/3 = 5 - 1/3$$
$$z = 4\ 2/3$$

The Use of Opposites and Reciprocals in Solving Equations

Consider: $2x - 1 = -3$

To put this equation in a simpler form, we first of all add the opposite of -1; that is +1, to both sides:

$$2x - 1 + 1 = -3 + 1$$
$$2x = -2$$

Now this equation can be solved by reciprocals. Since the reciprocal of 2 is 1/2, we have:

$$1/2 \cdot 2x = 1/2 \cdot -2$$
$$x = -1$$

Now consider: $\frac{2x}{5} + 3 = 6$

To reduce this equation to a simpler form, we first of all add the opposite of 3; that is -3, to both sides:

$$\frac{2x}{5} + 3 - 3 = 6 - 3$$
$$\frac{2x}{5} = 3$$

Since the reciprocal of 2/5 is 5/2, we have

$$\frac{5}{2} \cdot \frac{2}{5x} = \frac{5}{2} \cdot 3$$

$$x = \frac{15}{2}$$

In general, to solve this type of equation one should add the opposite and then multiply the reciprocal.

Exercise 3: Solve each equation:

(a) $-3x + 7 = -4$ (b) $3x/5 - 2 = 7$ (c) $-x/7 + 8 = -3$

Solutions:

(a) $-3x + 7 = -4$

$-3x + 7 - 7 = -4 - 7$ (adding -7 to both sides)

$-3x = -11$

$(-1/3)(-3x) = (-1/3)(-11)$ multiplying both sides by -1/3

$x = -11/3$

(b) $3x/5 - 2 = 7$

$3x/5 - 2 + 2 = 7 + 2$ adding 2 to both sides

$3x/5 = 9$

$5/3 \cdot 3x/5 = 5/3 \cdot 9$ multiplying both sides by 5/3

$x = 15$

(c) $-x/7 + 8 = -3$

$-x/7 + 8 - 8 = -3 - 8$ adding -8 to both sides

$-x/7 + 8 - 8 = -3 - 8$ adding -8 to both sides

$-x/7 = -11$

$(-7)(-x/7) = -7(-11)$ multiplying both sides by -7

$x = 77$

Other Equations

Here we shall present examples of more complicated linear equations and their step-by-step solutions. Each example will be simplified to a form previously seen.

Example 1: Solve $7x + 8x - 2x = 3 + 7 - 2$

First we must combine the like terms:

$$7x + 8x - 2x = 13x \text{ and } 3 + 7 - 2 = 8$$

Therefore, we have

$$13x = 8$$
$$x = 8/13 \text{ after multiplying both sides by } 1/13 .$$

Example 2: Solve $9x - 3 - 2x + 7 = 8 - 2$

Combining like terms, we have

$$9x - 2x - 3 + 7 = 8 - 2$$
$$7x + 4 = 6$$

This form we have seen before. Therefore

$$7x + 4 - 4 = 6 - 4 \text{ adding } -4 \text{ to both sides}$$

$$7x = 2$$
$$x = 2/7 \text{ after multiplying both sides by } 1/7$$

Example 3: Solve $5x + 2 = 2x - 4$

This is the first time that we have seen an x-term on both sides of the equation. In this equation, we will eliminate 2x from the right side by adding -2x to both sides.

$$5x - 2x + 2 = 2x - 2x - 4$$
$$3x + 2 = -4$$

Proceeding

$$3x + 2 - 2 = -4 - 2 \text{ adding } -2 \text{ to both sides}$$
$$x = -2 \text{ after multiplying both sides by } 1/3$$

Example 4: Solve $3(x + 2) = 8$

To solve this type of problem we must first eliminate the parentheses by multiplying each term in them by the 3 which is outside. Therefore

$3(x + 2) = 8$

because $3x + 6 = 8$

thus $3x = 2$ by adding -6 to both sides

$x = 2/3$ by multiplying both sides by 1/3.

Example 5: Solve $7x - (2 + 3x) = 9$

First eliminate the parentheses. Noting $-(2 - 3x)$ is the same as $-1 \cdot (2 + 3)$, we have

$$7x - 2 - 3x = 9$$

Combining like terms we have

$4x - 2 = 9$

$4x = 11$ by adding 2 to both sides

$x = 11/4$ by multiplying both sides by 1/4.

Summary

To solve any linear equation, proceed as follows:

Eliminate all parentheses.

Put all terms involving the variable on one side (by adding opposites to both sides).

Put all constant terms on the other side (by adding opposites to both sides).

Combine all like terms on both sides.

Multiply both sides by the reciprocal of the coefficient of the variable.

Checking Your Solution

To solve an equation like $3x + 6 = 12$, you find the value of x which when substituted for x will make the left side equal to the right side.

Upon solving $3x + 6 = 12$, you get $x = 2$ as a solution. You check this solution as follows:

$$3(2) + 6 = 12$$

$$6 + 6 = 12$$

$$12 = 12$$

The two sides have the same value, so therefore x = 2 is a correct solution.

Example 1: Is 2/3 the solution of $3(x + 2) = 8$?

$3(\underline{2/3} + 2) = 3 \quad 2/3 + 3 \quad 2 = 2 + 6 = 8$. Therefore, the left side equals the right side when $x = 2/3$. This means that $2/3$ is a solution of $3(x + 2) = 8$.

Example 2: Is 2 a solution of $3y + 4y = 5(y + 1)$?

In order to be a solution to this equation, when 2 is substituted for y, the left and right sides must be equal. First consider the left side, when y = 2.

$$3(\underline{2}) + 4(\underline{2}) = 6 + 8 = 10$$

Now consider the right side, when y = 2

$$5(\underline{2} + 1) = 15$$

Since

$$3(\underline{2}) + 4(\underline{2}) = 5(\underline{2} + 1)$$

the check fails, and as a result 2 is <u>not</u> a solution of $3y + 4y = 5(y + 1)$.

Example 3: Solve and check $5x + 2 = 7$

Solution:

$$5x + 2 = 7$$

$$5x + 2 - 2 = 7 - 2$$

$$5x = 5$$

$$x = 1$$

Check:

$$5(1) + 2 = 7$$

$$5 + 2 = 7$$

$$7 = 7$$

This verifies that x = 1 is a solution of $5x + 2 = 7$.

Word Problems

Before you will be able to solve a word problem you must first be able to convert English phrases into algebraic expressions.

Example: Write algebraic expressions to represent:

(a) three more than y Answer: $y + 3$

(b) eight less than x Answer: $x - 8$

(c) four less than twice x Answer: $2x - 4$

(d) twice the sum of six and three times x
Answer: $2(3x + 6)$

(e) the number of cents in x nickels
Answer: Since there are five cents in a nickel, then in x nickels the number of cents you have is $5x$.

(f) seven less than four times y is twice y
Answer: $4y - 7 = 2y$

(g) nine more than x divided by two is three more than five times x
Answer: $x/2 = 9 = x + 3$

(h) twice the sum of three times x and ten is two less than x
Answer: $2(3x + 10) = x - 2$

Now we will look at some examples of word problems.

Example 1: If the sum of two numbers is 12, and the larger number is three less than two times the smaller number, what are the values of the two numbers?

Solution: To solve a problem of this nature you must first of all keep in mind what is being asked for. Here you want to find the value of the two numbers -- one of which is smaller than the other. Now we must represent these numbers symbolically.

Let x = the smaller number.

Since we know that the sum of the two numbers is 12, we have the following equation:

$$x + 2x - 3 = 12$$

Combining terms, we have

$$3x - 3 = 12$$
$$3x = 15$$
$$x = 5$$

As a result, the smaller number = $x = 5$
the larger number = $2x - 3 = 2(5) - 3 = 7$

Example 2: The sum of four consecutive numbers is 34. Find the numbers.

Solution: Let x = the smallest number (first)
x + 1 = the second number
x = 2 = the third number
x = 3 = the fourth number

The sum of the numbers is 34. Therefore

$$x + (x + 1) + (x + 2) + (x + 3) = 4x + 6 = 34$$
$$= 4x + 6 - 6 = 34 - 6$$
$$4x = 28$$
$$x = 7$$

Therefore, the first number = x = 7
the second number = x + 1 = 8
the third number = x + 2 = 9
the fourth number = x + 3 = 10 ,

and upon adding these numbers you get the sum of 34.

Now let us consider an example of a somewhat more complicated type of word problem -- the mixture problem.

Example 3: We are going to mix 3 pounds of candy worth 60¢ per pound with 2 pounds of candy worth 40¢ per pound. What should the price per pound of the mixture be?

Solution: The solution of a mixture problem is based on the fact that the cost of the mixture will be equal to the cost of the individual ingredients. Given two ingredients,this can be written as

(cost of mixture = (cost of ingredient I) + (cost of ingredient II)).

In this problem

(cost of ingredient I) = (3 lbs.) · (60¢/lb) = 3 · 60¢ = 180¢

(cost of ingredient II) = (2 lbs) · (40¢/lb) = 2 · 40¢ = 80¢

Note that 60¢/lb means "60¢ per pound", and if we have 3 lbs of candy at 60¢/lb then it will cost 180¢. The number of pounds in the mixture will be the sum of the number of pounds in each ingredient. In this case, it is 2 lbs. + 3 lbs = 5 lbs. Since we don't know the price per pound of the mixture we will call it x. Therefore,

(cost of mixture) = 5x

and as a result:

$$5x = 180 + 80$$
$$5x = 260$$
$$x = 260/5 = 52 .$$

Now recall that x equals cents per pound. Therefore, the cost of the mixture is 52¢ per pound.

The above problem can be easily organized in the following table where the units are pounds.

	Number of pounds	Price per pound	Value
Ingredient I	3	60¢	(3)(60)
Ingredient II	+ 2	40¢ +	(2)(40)
Mixture	5	x	5x

From the last column, we obtain the equation

$$5x = 3 \cdot 60 + 2 \cdot 40$$
$$5x = 180 + 80$$
$$5x = 260$$
$$x = 52¢ \text{ per pound}$$

Note this is the same result as above. In general, any mixture problem can be put into a table like this one, but instead of pounds you will use whatever units are called for in the particular problem.

When solving word problems, the following steps should be followed in order:

Step 1: Set up the representation of the unknowns.

Step 2: Form an equation.

Step 3: Solve the equation.

Exercise 1: Solve the following problems:

(a) A twenty-one foot board is cut into two parts. If the smaller part is three feet less than the larger part, what is the length of each part?

Answer: 12 ft., 9 ft.

(b) Joe is four years older than Mary. Seven years ago, he was twice as old as she was then. How old is each today?

Answer: Joe is 15 years old today; Mary is 11 years old today.

(c) We are going to mix 20 pounds of candy worth 85¢ per pound with 10 pounds of candy worth 70¢ per pound. What should the price per pound of the mixture be?

Answer: 80¢ per pound

Ratio and Proportion

Any fraction can be thought of as a comparison of two numbers; that is, a ratio. For example, 3/4 compares 3 with 4. It is called the ratio of 3 to 4, and can be written 3/4 or 3:4. For example, the fraction 2/5 can be written as the ratio of 2 to 5 on the one hand, and on the other hand, the ratio 2 to 5 can be written as the fraction 2/5.

Exercise 1: Fill in the following table:

Fractions	Ratios
6/7	3 to 2
3/7	9 to 13
4/5	32 to 97

If Sam walks 3 miles in the time that Joe walks 8 miles, then

3/8 is the ratio of Sam's walking to Joe's walking

8/3 is the ratio of Joe's walking to Sam's walking

In this example, if Joe walks 16 miles, then Sam walks 3/8 as much as 16

Sam's walking = 3/8 x 16 = 6 miles

Similarly, if Sam walks 9 miles, then Joe walks 8/3 as many as 9

Joe's walking = 8/3 x 9 = 24 miles

A proportion is an equation in which both sides are fractions. Whenever you convert a fraction to one with a different denominator, you are working with a proportion. If, for example, you change 2/3 to sixths, then you have

2/3 = 4/6

This is an example of a proportion, and it can be said that the ratios 2:3 and 4:6 form a proportion. Other examples of proportions are:

1/2 = 4/8

4/5 = 20/25

6/18 = 1/3

Consider the following figures:

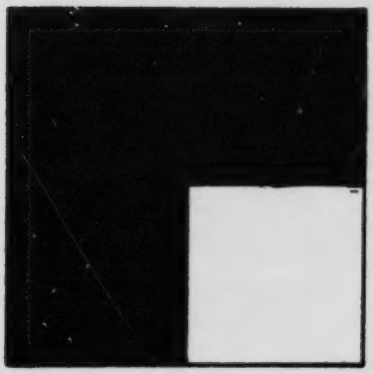


Figure 3

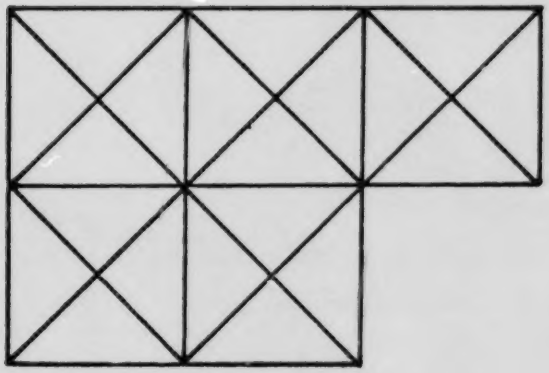


Figure 4

How many triangles in Figure 4 must be shaded so that the same proportion of the figure is shaded as in Figure 3 ? (Assuming all the triangles of Figure 4 are of equal area).

Solution:

3 of the 4 squares in Figure 3 are shaded in. Therefore 3/4 of this figure is shaded. Let x be the number of triangles in Figure 4 that must be shaded in. Since there are 12 triangles in Figure 4, then x/12 of this figure is to be shaded. Therefore, we must solve the following proportion to get our result:

$$x/12 = 3/4$$

$$x = 3/4 \quad 12 = 9$$

Therefore, 9 triangles in Figure 4 must be shaded in.

This reveals another example of a proportion. Further examples are:

$$6/7 = x/9$$

$$n/12 = 6/5$$

$$2/9 = 3/y$$

The following property can be used to determine the missing number in a proportion:

Product of extremes = Product of means

In the proportion $4/10 = 2/5$, the extremes are 4 and 5 and the means are 10 and 2. Applying this property to

$$4/10 = 2/5$$

we get

$$4 \cdot 5 = 2 \cdot 10 \text{ , which is true.}$$

Can you discover why this property always is true?

Now let us apply this property to

$$x/12 = 3/4$$

We have

$$\begin{aligned} 4 \cdot x &= 3 \cdot 12 \\ 4x &= 36 \\ x &= 9 \end{aligned}$$

Note this is the same result we obtained above.

Example 1: Find n

$$4/5 = 20/n$$

Using the fact that the product of the extremes equals the product of the means we obtain

$$\begin{aligned} 4 \cdot n &= 5 \cdot 20 \\ 4n &= 100 \\ n &= 100/4 = 25 \end{aligned}$$

Example 2: Find y

$$15/y = 5/6$$

Since the product of the extremes equals the product of the means, we have

$$\begin{aligned} 6 \cdot 15 &= 5y \\ 90 &= 5y \\ 5y &= 90 \\ y &= 18. \end{aligned}$$

This property is also useful in determining whether two ratios are proportional or not. Consider the ratios 2:3 and 6:9. Now 2:3 can be converted to the fraction 2/3 and 6:9 can be converted to the fraction 6/9. In order for 2/3 and 6/9 to be proportional, the product of the extremes must equal the product of the means. Checking this we have $2 \cdot 9 = 3 \cdot 6$, therefore we know that the ratios 2:3 and 6:9 are proportional, that is 2/3 = 6/9.

Example 3: Do 2:7 and 4:14 form a proportion?

Solution: 2: 7 can be converted to 2/7
4:14 can be converted to 4/14

Consider 2/7 and 4/14

Product of extremes = $2 \cdot 14 = 28$ Product of means = $7 \cdot 4 = 28$

Since they are equal, 2:7 and 4:14 are proportional; that is 2/7 = 4/14.

Exercise: Demonstrate whether the following ratios are proportional.

(a) 3:6 and 12:24

(b) 7:13 and 16:26

(c) 11:12 and 13:36

ALGEBRA OF EQUATION SOLVING: PART II

A second degree equation in one variable is an equation in which at least one term is of the second degree, but no term of higher degree appears. Commonly, these equations are called quadratic equations.

Examples:

(a) $5x^2 - 7x + 2 = 0$

(b) $3x^2 = 2x + 1$

Every quadratic equation in one variable can be put in the form

$$ax^2 + bx + c = 0$$

where a, b, and c are specific numbers in any given situation. For example, in 1 above $a = 5$, $b = -7$ and $c = 2$. $ax^2 + bx + c = 0$ is often called the standard form for a quadratic equation in one variable. The letters b or c or both may be zero but an x may never be zero. You can always identify a quadratic expression in one variable if you keep in mind that it is always an equation of second degree that can be set equal to zero. For example, 2 above can be expressed in the standard form if we transpose all the terms to the left side of the equation. $3x^2 = 2x + 1$ becomes $3x^2 - 2x - 1 = 0$ which is now in standard form with $a = 3$, $b = -2$ and $c = -1$.

Much to our delight, a general formula which can easily be applied to solve any quadratic equation has been established. The equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which really means

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

gives the solution to any second degree equation and is called the quadratic formula.

Examples:

1. Solve the equation $x^2 - 5x + 6 = 0$ by the quadratic formula and check by substitution.

Solution: Since the equation is in the proper form, we identify $a = 1$, $b = -5$, and $c = 6$. Therefore

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{25 - 24}}{2}$$

$$= \frac{5 \pm 1}{2}$$

Hence,

$$x = \frac{5 + 1}{2} = 3 \quad \text{and} \quad x = \frac{5 - 1}{2} = 2$$

To check:

$(3)^2 - 5(3) + 6 = 0$
$9 - 15 + 6 = 0$
$-6 + 6 = 0$
$0 = 0$

and

$(2) - 5(2) + 6 = 0$
$4 - 10 + 6 = 0$
$-6 + 6 = 0$
$0 = 0$

2. Solve the equation $5x^2 = 15x$ by the quadratic formula and check by substitution.

Solution: Before the formula can be used the equation must be put in standard form. In so doing we get

$$5x^2 - 15x = 0$$

Therefore, $a = 5$, $b = -15$ and $c = 0$.

The solutions are:

$$x = \frac{-(-15) + \sqrt{(-15)^2 - 4(5)(0)}}{2(5)}$$

$$= \frac{15 \pm \sqrt{225}}{10}$$

$$= \frac{15 \pm 15}{10}$$

Hence,

$$x = \frac{15 + 15}{10} = 3 \quad \text{and} \quad x = \frac{15 - 15}{10} = 0$$

To check:

$5(3)^2 = 15(3)$ and $5(0)^2 = 15(0)$

$5(9) = 45$ and $0 = 0$

$45 = 45$

Exercises:

1. Write the quadratic equation from memory.
2. Use the quadratic equation to solve the following equations:
   (a) $x^2 - 7x + 10 = 0$
   (b) $2x^2 + 5x - 3 = 0$
   (c) $3x^2 - 12 = 0$
   (d) $x^2 + 4 = 5x$
   (e) $9x^2 - 5x = 0$
   (f) $4x^2 = 36$

Answers:

(a) $x = 2,\ x = 5$

(b) $x = 1/2,\ x = -3$

(c) $x = \pm 2$

(d) $x = 4,\ x = 1$

(e) $x = 5/9,\ x = 0$

(f) $x = \pm 3$

Application

Not all problems that can be translated into algebraic equations are of the first degree. Some problems give rise to equations of higher degree. In this section you will learn to form second degree equations in one variable. Then you may apply the quadratic formula that you learned from the preceding section to solve your equations.

As you have often seen, often problems aren't presented in the form of equations, particularly word problems that depict practical situations. If we are to solve them mathematically, we must first translate them into equations. You are already familiar with this process, however, having used it earlier when you worked with equations in the first degree. The approach is basically the same:

Suppose you are given that three times a positive number subtracted from the square of the number results in 4. How would you translate this statement into a second degree quadratic equation? Read the problem carefully and think about what is being asked. After giving it thought, write an accurate statement of what the unknown variable is to represent. Let x = the positive number. In the next step we express what is given in terms of the unknown variable, i.e.:

$$x^2 = \text{the number squared}$$

$$3x = \text{the number subtracted}$$

Now we will write an algebraic equation that expresses the problem in terms of the unknown. Hence:

$$x^2 - 3x = 4$$

Now solve the equation and check your results in the original statement of the problem. Let's do that by using the quadratic formula. Putting the equation in standard form, we get

$$x^2 - 3x - 4 = 0, \quad a = 1, \quad b = -3, \quad \text{and} \quad c = -4$$

Therefore

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{25}}{2}$$

$$= \frac{3 + 5}{2}$$

So

$$x = \frac{3 + 5}{2} \rightarrow 4 \quad \text{and} \quad x = \frac{3 - 5}{2} \rightarrow -1$$

Since in the original problem the number is described as being positive, we take x = 4 as our answer. Both answers are mathematically correct, however.

Now to check, if we subtract $3(4) = 12$ from $x^2 = (4)^2 = 16$, we do indeed get 4, i.e.:

$$x^2 - 3x = 4$$

$$(4)^2 - 3(4) = 4$$

$$16 - 12 = 4$$

$$4 = 4$$

Let's look at another example:

A father is twice the age of his daughter; the square of her age is 10 times his age. How old is each?

Solution: Let x = the daughter's age

$2x$ = the father's age

x^2 = the daughter's age squared

Then $x^2 = 10(2x)$

And $x^2 = 20x$

Changing to standard form:

$$x^2 - 20x = 0$$

is the equation we want. Using the quadratic formula we get:

$$x = \frac{-(20) \pm \sqrt{(-20)^2 - 4(1)(0)}}{2(1)}$$

$$= \frac{20 \pm \sqrt{400}}{2}$$

$$= \frac{20 \pm 20}{2}$$

$$x = \frac{20 + 20}{2} = 20 \quad \text{and} \quad x = \frac{20 - 20}{2} = 0$$

We conclude that the daughter is 20 years old and her father is 40.

To check:

$$x^2 = 20x$$

$$(20)^2 = 20(20)$$

$$400 = 400$$

Exercises:

Write quadratic equations for each of the following and solve each equation by using the quadratic equation.

1. Three times the square of an integer is twice the integer plus one. Find the integer.

2. A certain number is 42 less than its square. Find the number.

3. Two numbers differ by 6. Their product is 91. Find the number.

4. The difference between two integers is 6. If their product is -8, find the numbers.

Answers:

1. 1
2. 7, -6
3. 13, 7 or -13, -7
4. 2, 4 or -2, -4

Post-Test

1. Solve for x: $-4/5x = 13$
2. Solve for y: $6y - 2 = 8$
3. Solve for x: $7x - 2 = 4x + 5$
4. Solve for n: $9(x + 3) = 18$
5. Solve for x: $3/7 = 8/x$
6. If an eleven foot long string is cut into two pieces and the larger piece is five less than three times the smaller piece, what is the length of each piece?
7. Do 5:6 and 15:18 form a proportion? Why or why not?
8. Solve for x by using the quadratic formula:

$$-2x^2 + 4x + 1 = 0$$

Write the following English words into an equation and solve:

9. Nine more than five times x is four times x.
10. Six less than x plus x^2 is equal to zero.

GRAPHS AND THEIR INTERPRETATION

Introduction

This unit deals with the use of graphs to represent and interpret situations, data, and mathematical formulas. The use of pictorial representation is becoming more widespread in the various branches of science, industry and government; thus, you must become proficient at constructing and reading graphs. You will find as you complete this section that a graph can be worth a thousand words and a million numbers.

Objectives

When you have completed this section, you should be able to do the following:

(1) Construct appropriate line, bar or circle graphs which represent given sets of data.

(2) Answer simple, interpretative questions related to given line, bar or circle graphs.

(3) Plot on rectangular graph paper any set of five points whose coordinates are given.

(4) Construct the graph of any first degree equation in two variables (line) by plotting and connecting at least three points.

(5) Construct the graph of any second degree equation in two variables (conic) by plotting and connecting at least five points.

(6) Find the coordinates of the point of intersection of any two nonparallel lines by the graphical method, correct to the nearest half of a unit.

Pre-Test

1. In a survey of 200 students, it was found that:

   100 students came to school by bus,
   25 students came by car, and
   75 students walked to school.

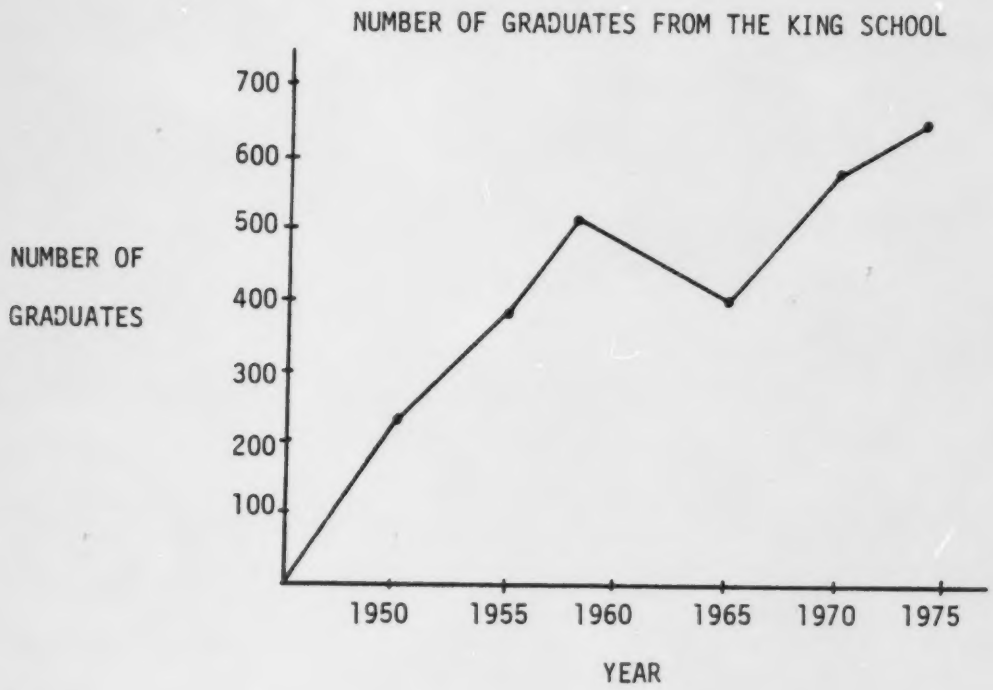
   Construct a circle graph to represent this information.

2. A mathematics test was given and the following results recorded:

10% of the students scored between 20 and 30,
27% scored between 30 and 40,
43% scored between 40 and 50, and
20% scored between 50 and 60.

Make a bar graph to show the above information.

3.



(a) In what years did the number of graduates at King High School show a decrease?

(b) Approximately how many more students graduated in 1970 than in 1950?

(c) To the nearest hundred, how many students graduated in 1957? How many in 1973?

4. Plot the following points on a rectangular coordinate system:

(-1, 4), (3, 2), (5/2, -2), (-1, -3), (1/2, - 3/2)

5. Plot at least 4 points on the graph of $x^2 + y^2 = 36$ and sketch the graph.

6. Use the graphical method to determine the point of intersection of the two lines:

$$4x + y = 1$$

$$x - 2y = 4$$

Key Terms

graph
coordinate
axis
rectangular coordinate system
plot
origin

Study Materials

Circle Graphs

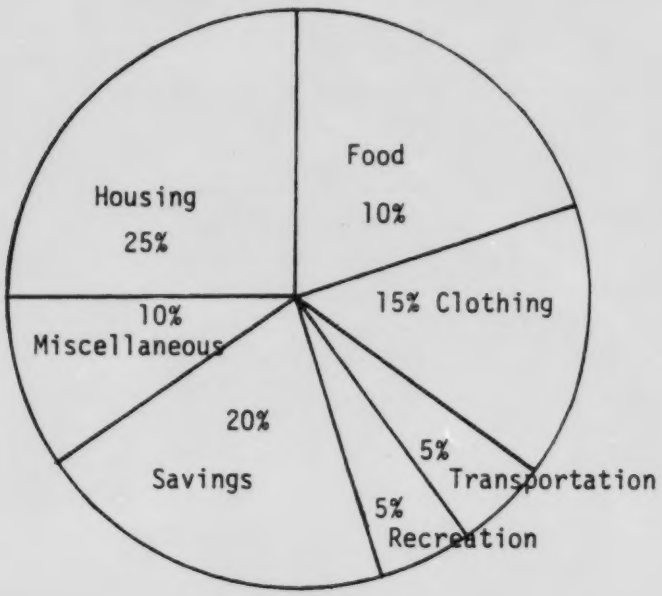
A circle graph or pie graph is a circular representation of a quantity in which parts of the quantity are represented by segments of a circle.

Example 1:

A family has an income of $800 per month. Their budget for a certain month is: housing $200; food $160; clothing $120; transportation $40; recreation $40; savings $160; miscellaneous $80.

What percentage of the total monthly income is each item above? Answer: housing 25%; food 20%; clothing 15%; transportation 5%; recreation 5%; savings 20%; miscellaneous 10%. A circle graph denoting the parts and percentages of the budget is given below.

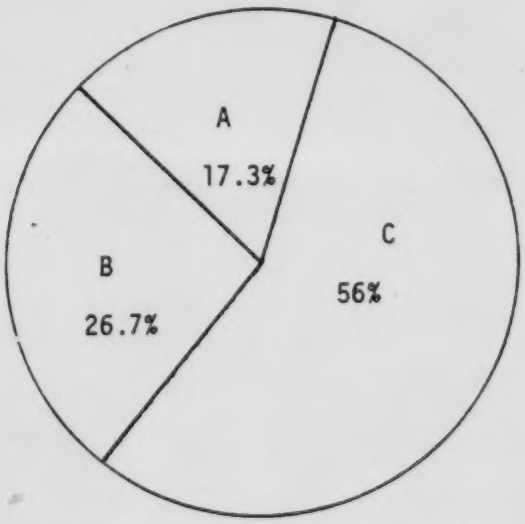
FAMILY BUDGET



Note that the 25% of the budget alloted for housing is represented by a segment of the circle which has an angle that is 25% of the total degrees in a circle; namely, $.25 \times 360^{\circ} = 90^{\circ}$. Similarly, the 20% associated with food is represented by a segment which has an angle of $.20 \times 360^{\circ} = 72^{\circ}$. Determine the angles contained in the remainder of the segments. Answers: $54^{\circ}, 18^{\circ}, 72^{\circ}, 36^{\circ}$.

Example 2:

The annual income report of a manufacturing company which produces items A, B, and C is given in the circle graph below. (The graph indicates the percentage of the total income generated by each item.)



(a) Which of the items produced the largest income? Answer: C.

(b) If $24,100 were produced by item A, what was the total income of the company? Answer: $139,307

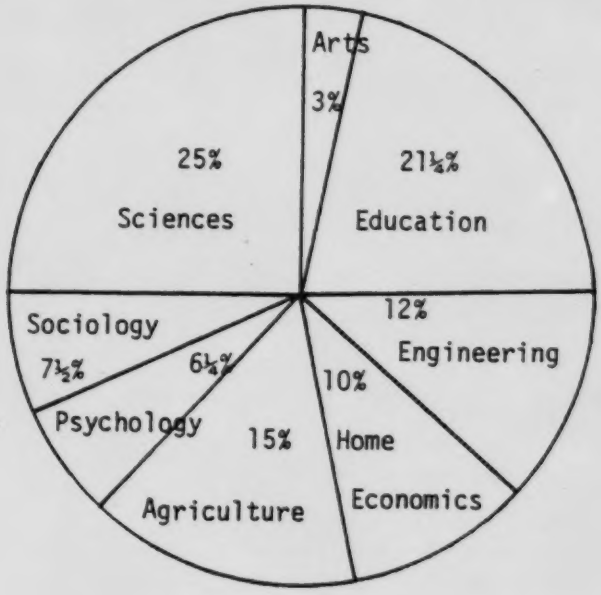
(c) How much income was generated by B? Answer: $37,195

Example 3:

At a certain college the distribution of major courses of study is: Sciences -100, Arts - 12, Education - 85, Engineering - 48, Home Economics - 40, Agriculture - 60, Psychology - 25, and Sociology - 30. Illustrate this distribution in a circle graph, and include the appropriate percentages.

Solution:

The total number of majors is 400. Consequently, the percentages for the various areas are: Science 25%, Arts 3%, Education 21 1/4%, Engineering 12%, Home Economics 10%, Agriculture 15%, Psychology 6 1/4%, Sociology 7 1/2%.

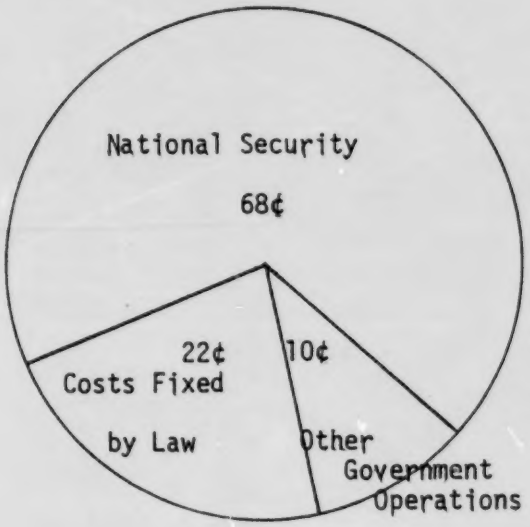


Exercises:

1. In a certain manufacturing firm it is decided that total expenditures will be broken down as follows: 10% advertising, 50% manufacturing costs, 20% salaries, and 20% miscellaneous. Make a circle graph which shows this breakdown.

2. If total expenditures in #1 are $1,500,000, what amount will be spent in each category?

3. The following graph indicates how the tax dollar was spent in the United States during a recent year:



(a) What percentage of total expenditures was for national security?

(b) Out of each $1,000 spent by the federal government, how many dollars were spent for national security?

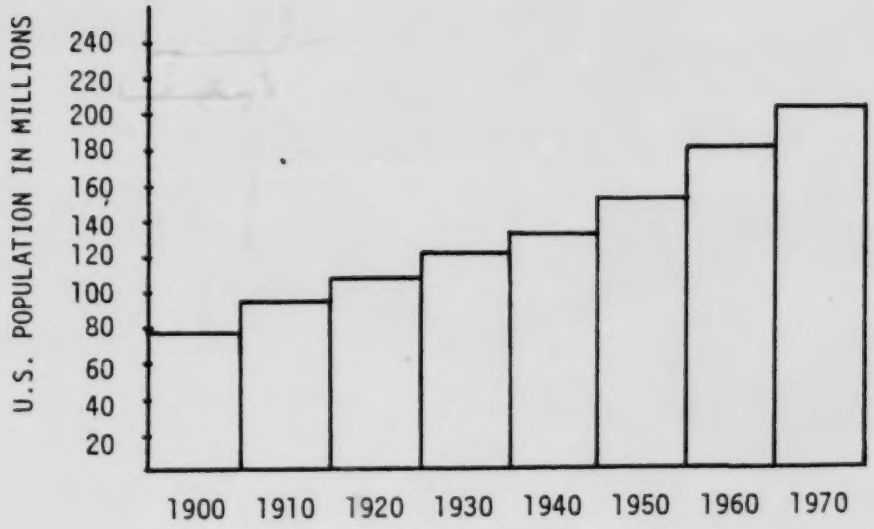
(c) If total expenditures were $800 billion, how much was spent in each category?

Bar Graphs

Bar graphs are normally used to compare amounts of a quantity relative to different time periods or relative to different categories. The graph consists of a series of rectangular bars of uniform width whose heights represent the various amounts of the quantity.

Example 1:

The bar graph below shows the population of the United States during each decade from 1900 - 1970.

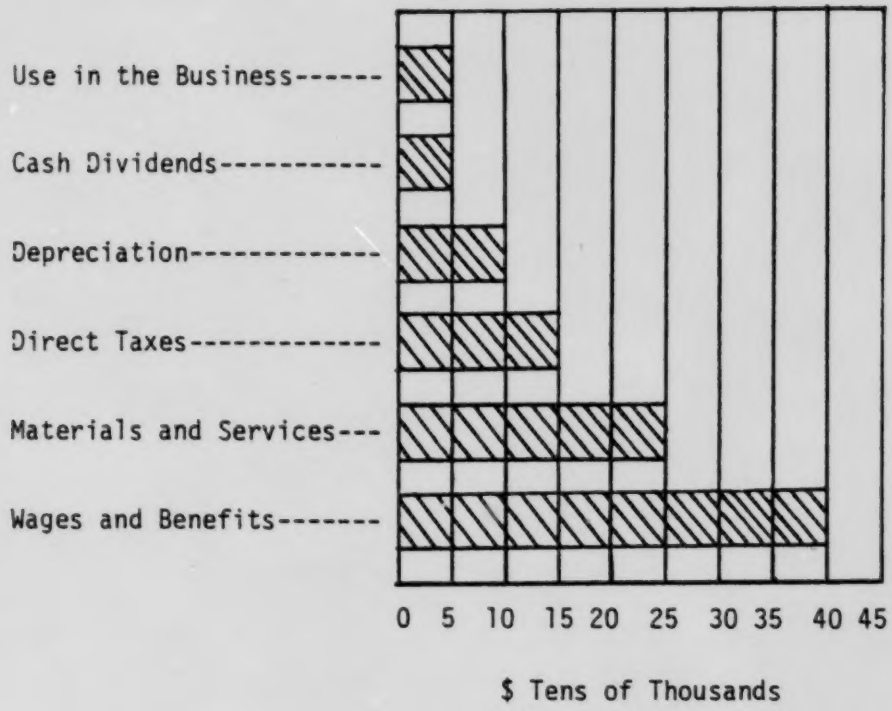


Use this bar graph to approximate the population of the U.S. in each of the given decades. Answers (in millions): 76.0, 92.0, 105.7, 122.8, 131.7, 151.1, 179.3, 203.2.

During which period did the U.S. population increase most rapidly? Which periods produced the slowest increase?

Example 2:

An industrial concern made expenditures during a recent year according to the following bar graph:



(a) What were total expenditures? Answer: $1,000,000

(b) How much was spent on direct taxes?

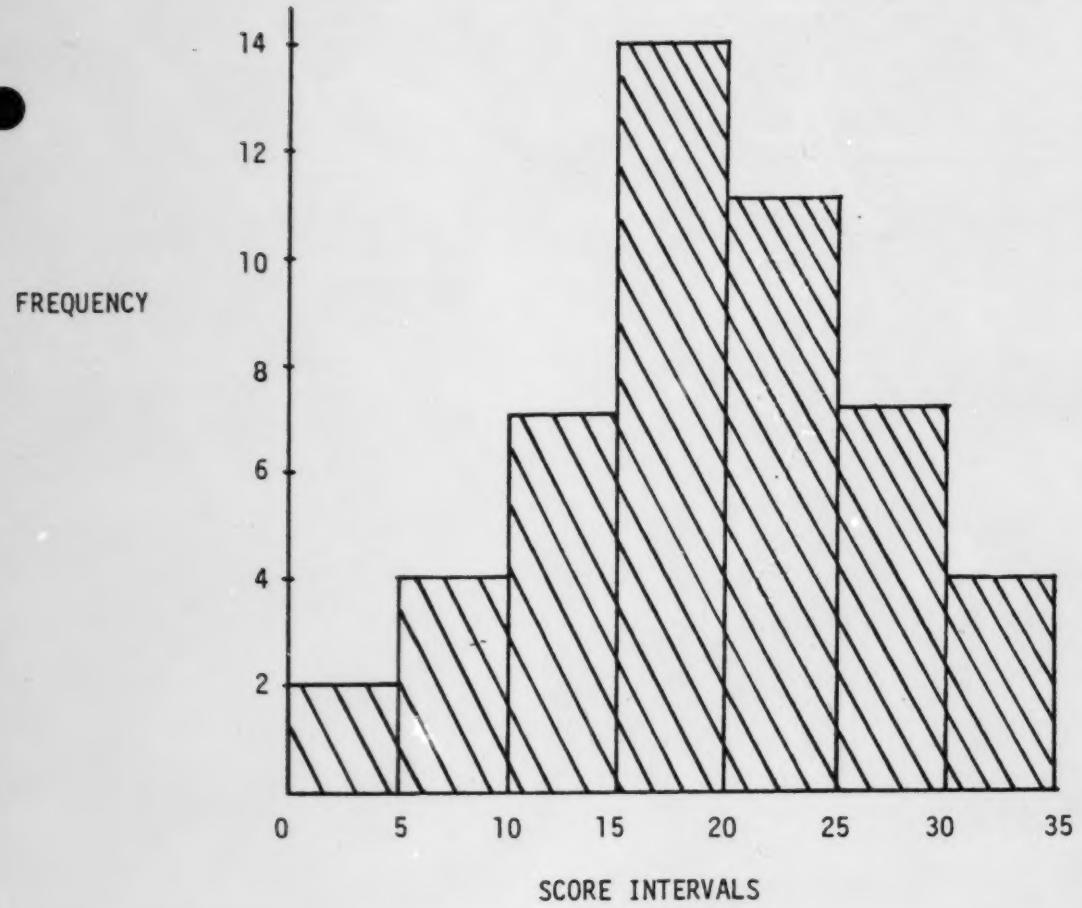
(c) What percentage of expenditures went toward wages and benefits?

Example 3:

The grades of 49 students on a math test were recorded in the chart below:

Score Intervals	0-5	5-10	10-15	15-20	20-25	25-30	30-35
Frequency	2	4	7	14	11	7	4

A bar graph corresponding to the chart is given below:



Exercises:

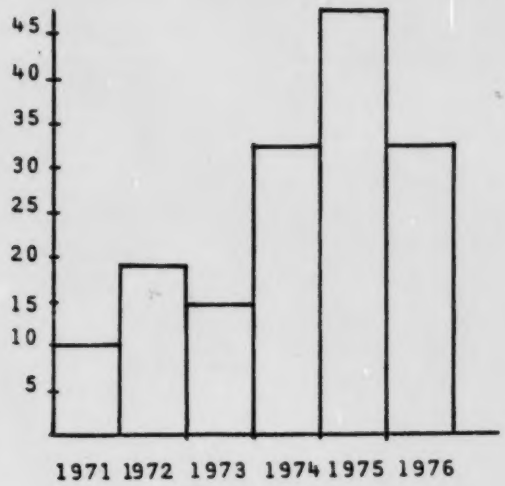
1. The graph below shows the amount of money earned by the Broadnax Corporation during 6 consecutive years.

   (a) What was the best year for earnings, and how much was earned? _____.

   (b) During which two consecutive years was the increase in earnings the greatest? _____

   (c) In which year was an increase in earnings succeeded by a decrease of the same amount during the following year? _____

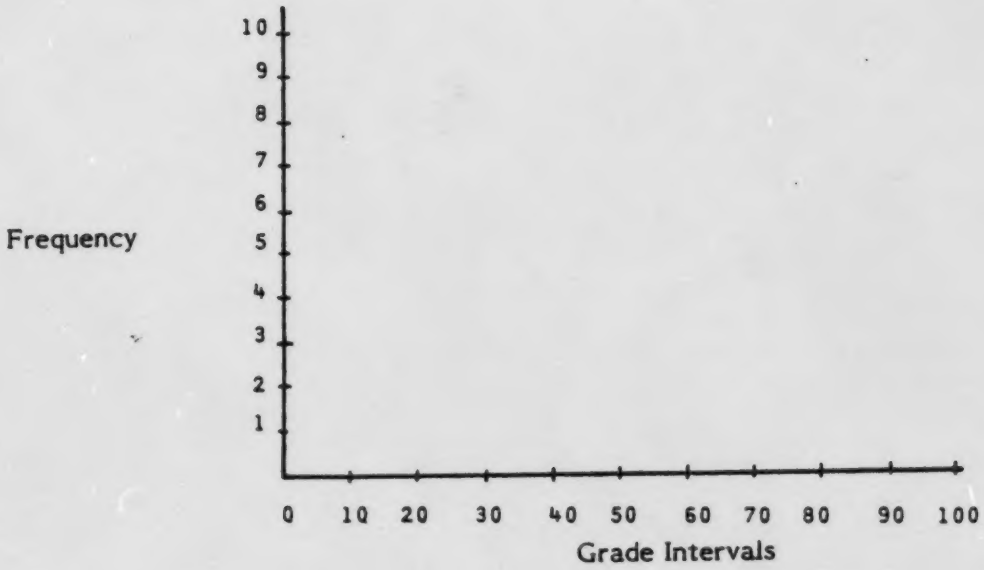
   (d) Give the amounts earned in each of the six years. _____



2. The following scores were recorded on a history test:

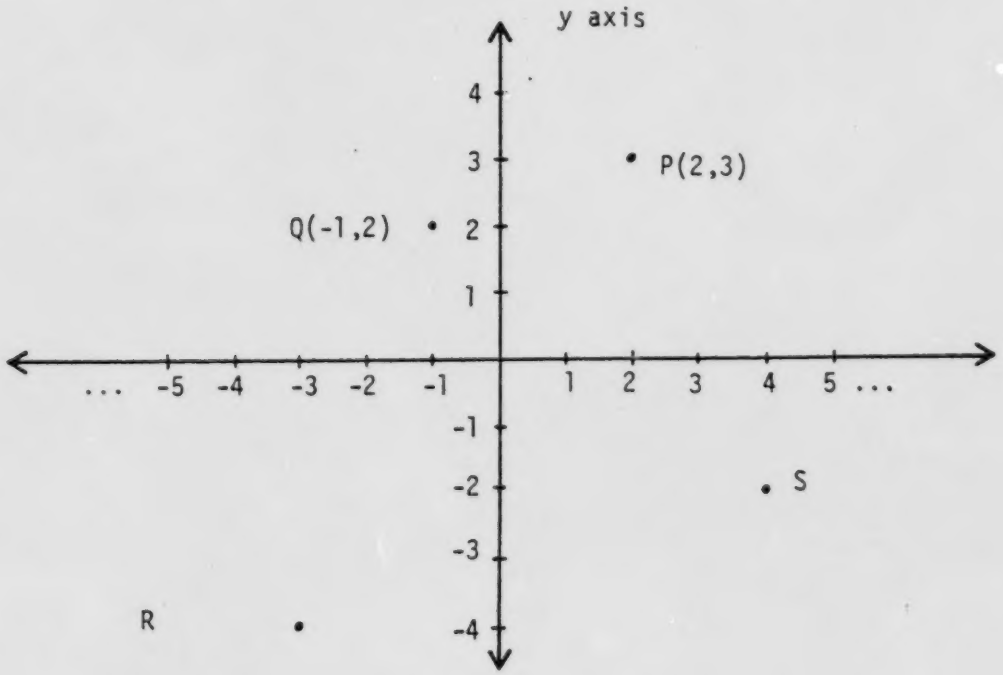
74, 18, 72, 20, 75, 45, 37, 80, 73, 51,
75, 68, 94, 92, 81, 79, 100, 69,
76, 83, 90, 48, 86, 70, 78.

Complete a bar graph of this data, using the chart in the figure below.



Line Graphs

Line graphs (or coordinate graphs) involve a rectangular coordinate system, as shown below, consisting of two intersecting straight lines called axes, scaled according to a specific unit size. The horizontal axis is often called the x-axis, and the vertical axis is called the y-axis, although these designations are not always used. One labels points in the plane according to their distances from the x and y axes. For example, the point P in the diagram is labeled (2, 3), since it is 2 units to the right of the y axis and 3 units above the x axis; the point Q is labeled (-1, 2), since it is 1 unit to the left of the y axis and 2 units above the x axis. Point P is said to have x coordinate 2 and y coordinate 3. What are the coordinates of points R and S in the diagram? Explain the meaning of these coordinates.



When you locate a point in the plane using a coodinate system, you are said to be plotting the point. When points are plotted and connected by lines, the resulting figure is referred to as a line graph on a coordinate graph.

The following set of exercises provides considerable practice in plotting points and line graphs.

Practice Problems

In 1, 2, and 3 below, the ordered pairs form a name or make a pattern. Plot the points in each case and connect them with lines in such a way that the result will depict one or more of the following:

Top, 007, GTO, Soul, Mark IV, Batman

1. The following set of ordered pairs forms a name or makes a pattern. Plot the points and try to guess the name or pattern.

(a)	(0,0)	(e)	(2, 3)	(i)	(4,3)
(b)	(5,2)	(f)	(-2, 3)	(j)	(0,3)
(c)	(-4, 3)	(g)	(6,3)	(k)	(2,0)
(d)	(-2, 0)	(h)	(4, 0)	(l)	(-4, 0)

It might help if you connect some of the points with straight lines.

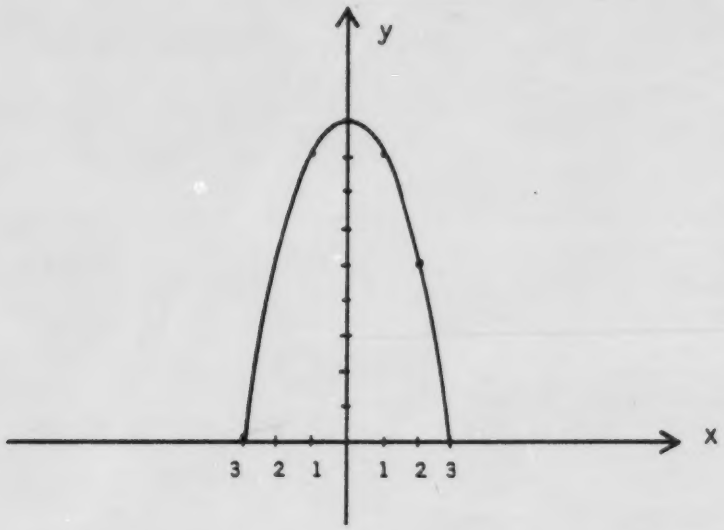
2. Try to guess the name or pattern formed by the following pairs:

(a)	(-4, 0)	(f)	(0,3)	(k)	(-1, 3)
(b)	(-2, 1½)	(g)	(-3, 1½)	(l)	(-2, 3)
(c)	(4, 0)	(h)	(4, 3)	(m)	(-4, 3)
(d)	(2,3)	(i)	(2,0)	(n)	(-2, 0)
(e)	(0, 0)	(j)	(1, 3)		

3. Do the same for:

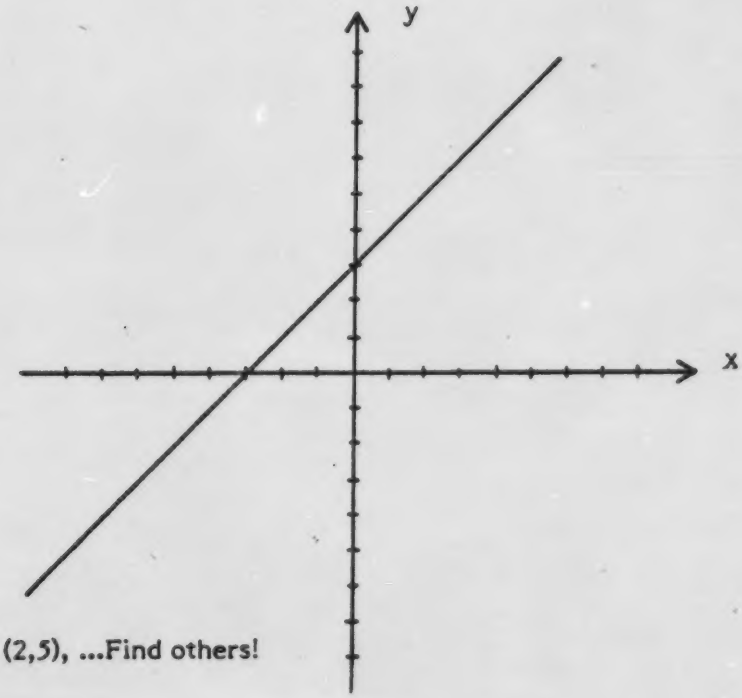
(a)	(3, 0)	(h)	(7,8)	(o)	(4,4)
(b)	(0, 3)	(i)	(8, 7)	(p)	(3, 3)
(c)	(4.5, 5)	(j)	(7, 11)	(q)	(5, 4)
(d)	(1, 7)	(k)	(9, 13)	(r)	(6, 3)
(e)	(6, 7)	(l)	(9, 3)	(s)	(6, 0)
(f)	(3, 7)	(m)	(0, 13)	(t)	(9, 3)
(g)	(2, 8)	(n)	(2, 11)	(u)	(4, 2)
				(v)	(5, 2)

4. Identify the coordinates of several points on the graph below.



Answers: (0,3), (1,4), (-3, 0), ... Find many others!!

5. Identify the coordinates of several points on the following graph: approximate when necessary.



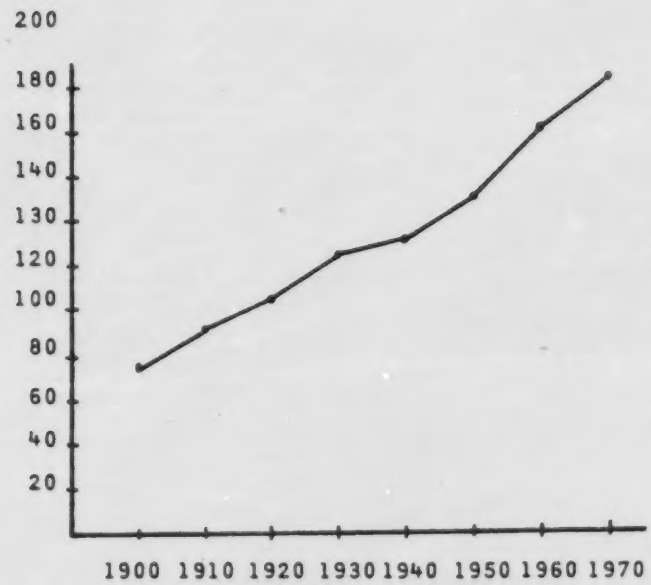
Answers: (1,8), (-1, 8), (2,5), ...Find others!

Graphs of Data

As an example of the use of line graphs, consider the population example of a previous section.

Year	1900	1910	1920	1930	1940	1950	1960	1970
Population (millions)	76	92	105.7	122.8	131.7	151.1	179.3	203.2

If we place years on the horizontal axis and population amounts on the vertical axis, a graph of this data can be obtained by graphing the "data points" (1900, 76), (1910, 92), etc.

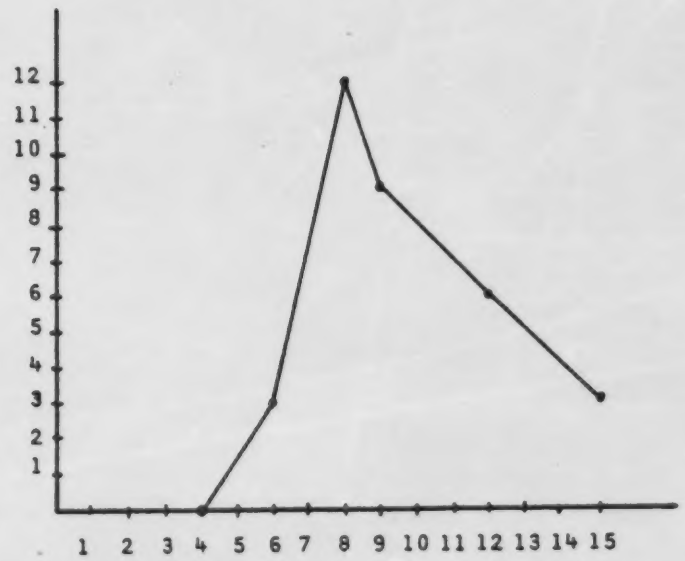


Example 1:

Construct a graph of the following data, which represent a set of scores on an examination.

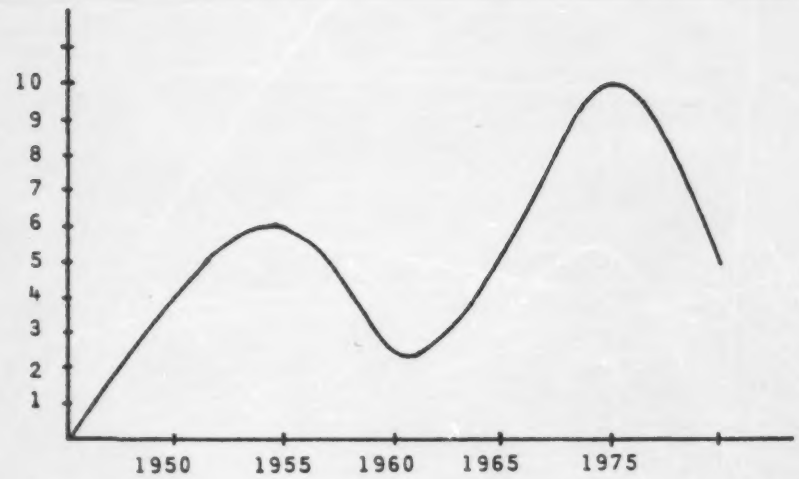
Score	Number of Students Making Score (Frequency)
15	3
8	12
9	9
12	6
6	3
4	0

Solution:



Example 2:

The following graph indicates the number of games won by a certain college football team each season over a 30-year period.



(a) What was the worst year for the team?

(b) What was the best year for the team?

(c) How many games did the team win in 1953?

(d) Over what 10-year period did the team perform best?

(e) Over what 10-year period did the team perform worst?

(f) What is the average number of games won per year over the 30-year period?

(g) What is the average number of losses per year over the 30-year period?

Exercise 1:

Graph the following data, which indicate the temperature of an object over a 12-hour period.

	Time	Temperature		Time	Temperature
	12	42°		2	73°
	2	40°		4	84°
A.M.	4	40°	P.M.	6	80°
	6	36°		8	60°
	8	52°		10	57°
	10	55°		12	40°
	12	62°			

Exercise 2:

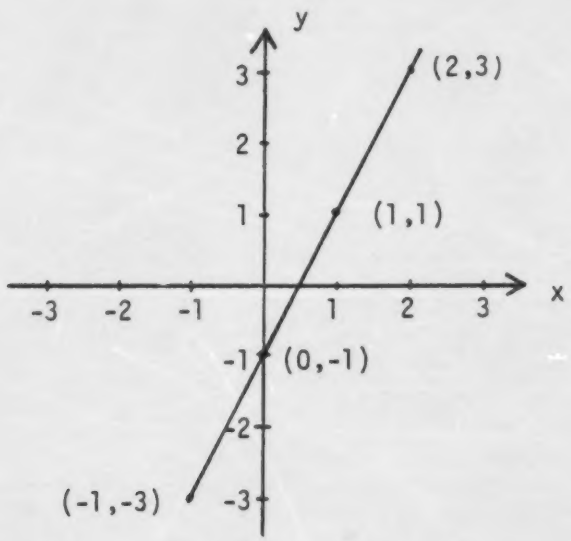
The chart below gives the average yearly Consumer Price Index (CPI) for a 10-year period, using the 1967 dollar as a basis for comparison. Draw a graph of this data.

Year	CPI
1962	90.6
1963	91.7
1964	92.9
1965	94.5
1966	97.2
1967	100.0
1968	104.2
1969	109.8
1970	116.3
1971	121.3

Graphs of First Degree Equations

Coordinate graphs are useful for obtaining "pictures" of equations in more than one variable. We have seen that a first degree equation in one variable has a single solution. But, a first degree equation in 2 variables has many solutions, and the plotting of these solutions produces a graph of the equation.

For example, the equation $y = 2x - 1$ has a solution $x = 1, y = 1$, since $1 = 2(1) - 1$. But $x = 2, y = 3$ is also a solution, as is $x = -1, y = -3$. Thus, a graph of the equation results from plotting the points (1, 1), (2, 3), (-1, -3), etc., and connecting the points as in the figure below.



Question: How can the coordinates of other points on the graph of this equation, $y = 2x - 1$, be found?

Answer:

Let x be any number, and place this value of x in the equation to determine the corresponding value of y. For example, if $x = 3$, then $y = 2(3) - 1 = 5$. Thus $x = 3, y = 5$ is a solution of the equation and the point (3.5) is on the graph.

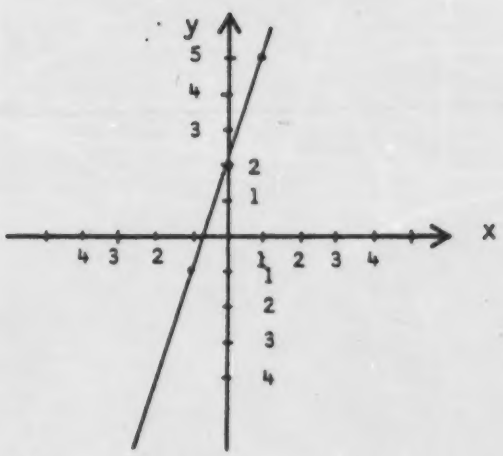
Use this method to complete the following table of solutions of the equation.

x	y
3	5
4	7
- 2	___
- 3	___
1/2	___

Example 1: Sketch a graph of the equation $y = 3x + 2$.

Solution:

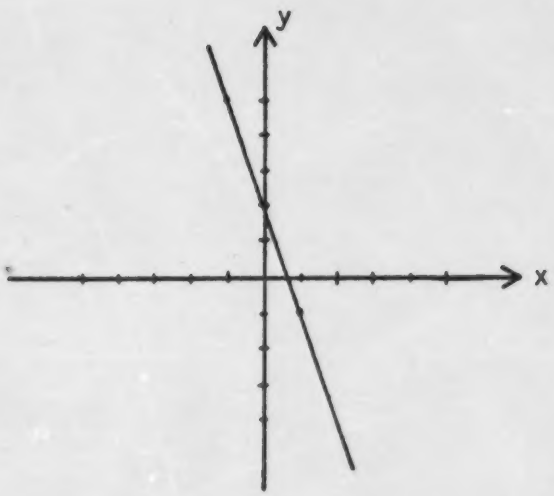
x	y
0	2
1	5
- 1	- 1



Example 2: Sketch a graph of the equation $2y = 4 - 6x$.

Solution:

x	y
0	2
1	-1
-1	5



These examples suggest the following well-known result:

A FIRST DEGREE EQUATION IN 2 VARIABLES HAS A GRAPH WHICH IS A STRAIGHT LINE.

Since two points determine a line, in order to graph a first degree equation in 2 variables it is necessary to plot only two solutions to the equation. But, as a check for accuracy, one should plot at least 3 solutions.

Exercises: Graph the following straight lines:

1. $y = 6 - 2x$

2. $3x + 4y = 2$

3. $\frac{x - y}{2} = 5$

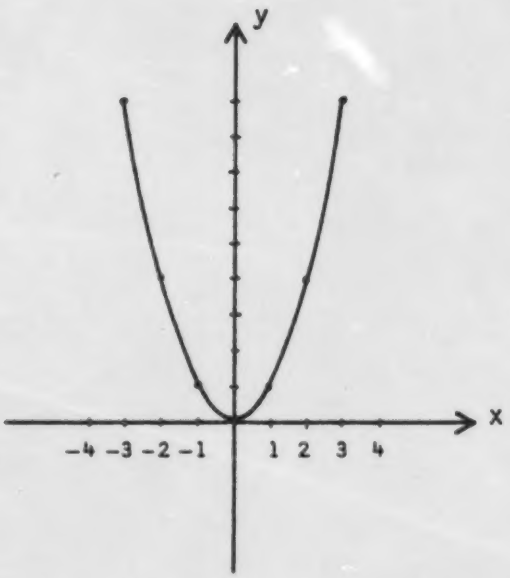
Graphs of Second Degree Equations

The method for graphing second degree equations in 2 variables is similar to the preceding.

Example 1: Sketch a graph of $y = x^2$.

Solution:

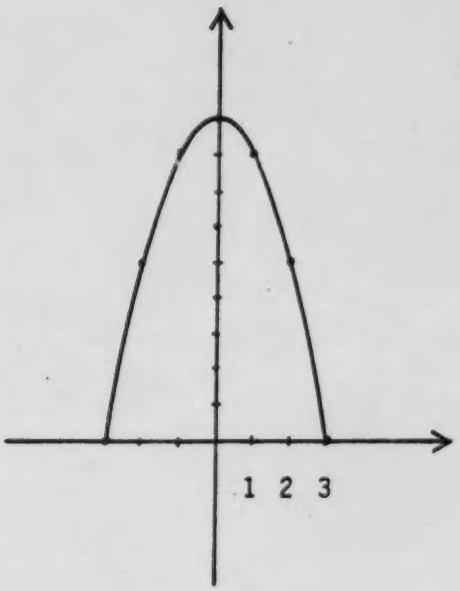
x	y
0	0
1	1
2	4
3	9
-1	1
-2	4
-3	9



Example 2: Sketch a graph of $y = 9 - x^2$

Solution:

x	y
0	9
1	8
2	5
3	0
- 1	8
- 2	5



The geometric figures in examples 1 and 2 are called parabolas.

Example 3: Sketch a graph of $x^2 + y^2 = 16$.

Solution:

Rewrite the equation by "separating" the variables:

$$y^2 = 16 - x^2 .$$

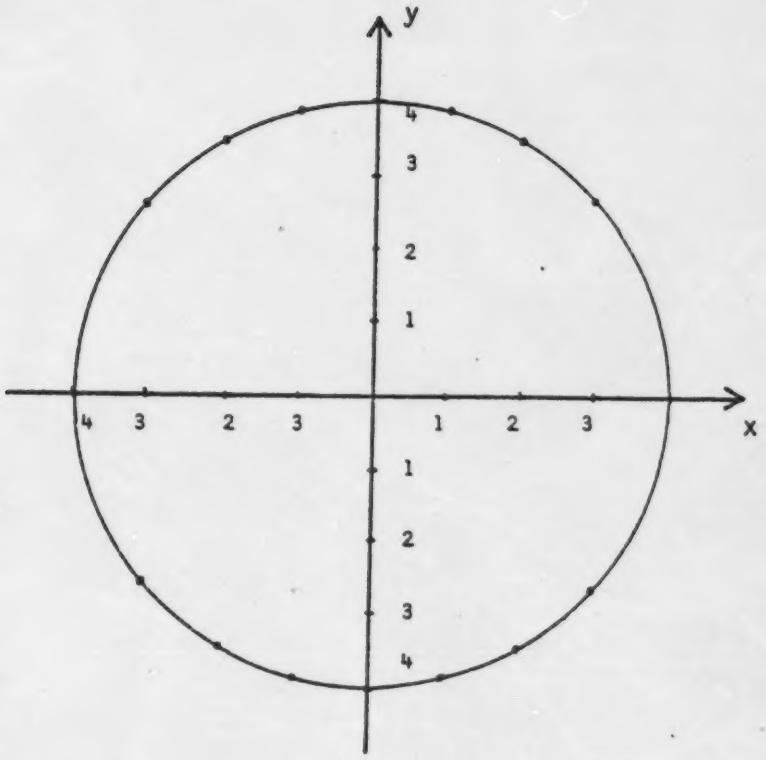
Now take the square of both sides:

$$y = \pm\sqrt{16 - x^2} .$$

Substituting values for x in this equation yields the following table of solutions:

We can now use approximations of the square roots to graph the points.

x	y
0	± 4
1	$\pm\sqrt{15}$
2	$\pm\sqrt{12}$
3	$\pm\sqrt{7}$
4	0
-1	$\pm\sqrt{15}$
-2	$\pm\sqrt{12}$
-3	$\pm\sqrt{7}$
-4	0



This resulting figure is a circle.

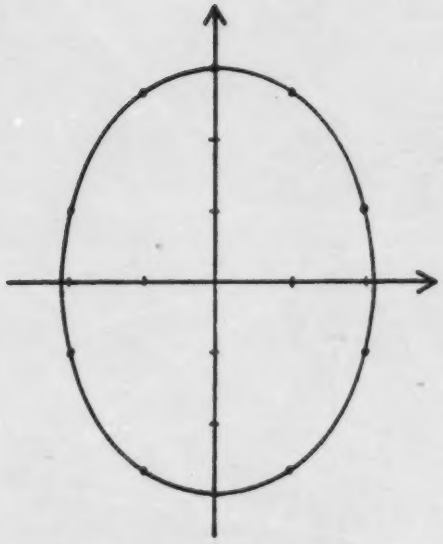
Example 4: Sketch a graph of $2x^2 + y^2 = 9$.

Solution:

$$y^2 = 9 - 2x^2$$

$$y = \pm\sqrt{9 - 2x^2}$$

x	y
0	$+3$
1	$\pm\sqrt{7}$
2	± 1
-1	$\pm\sqrt{7}$
-2	± 1



This is the graph of an ellipse.

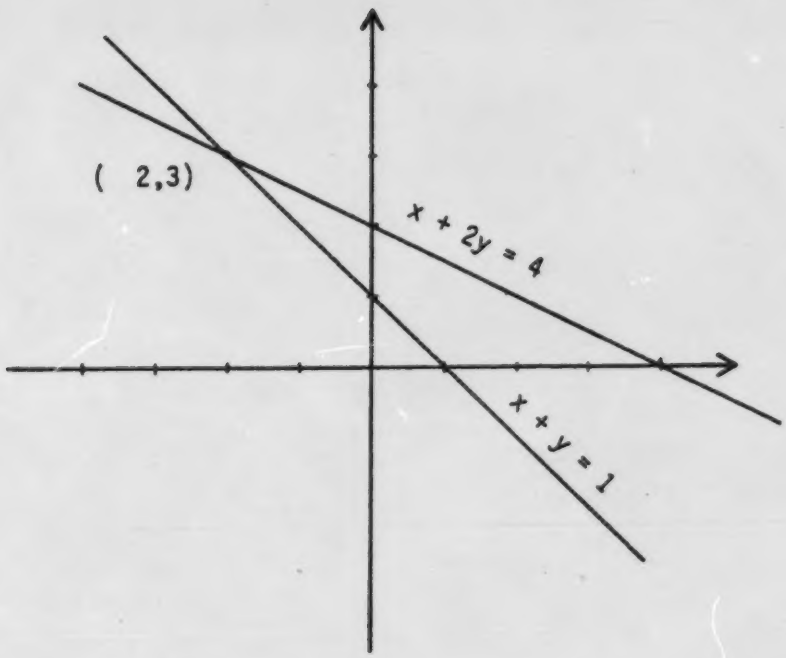
Example 1:

Find, by the graphical method, the solution of the following simultaneous equations:

$$x + y = 1$$

$$x + 2y = 4 .$$

Solution:



Answer: (-2, 3). Check the answer!

Exercises:

Plot graphs of the following equations:

1. $2x - 4y^2 = 1$ (parabola)
2. $x^2 + y^2 = 9$ (circle)
3. $x^2 + 4y^2 = 4$ (ellipse)
4. $x^2 - y^2 = 4$ (hyperbola)

Graphs of Simultaneous Equations

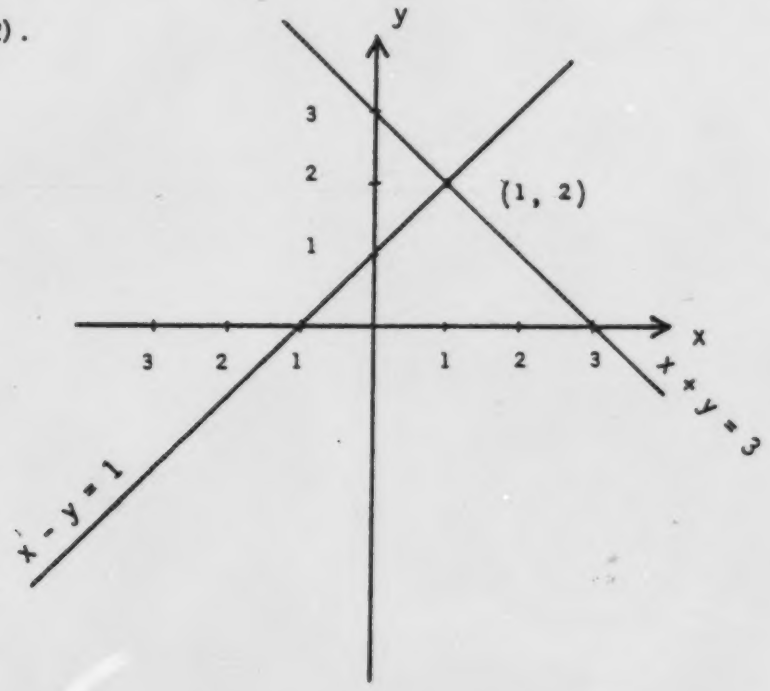
Draw the graph of each of the equations (lines) below on the same coordinate axis system.

$$x + y = 3$$

$$x - y = -1$$

Can you determine from your graph the coordinates of the point of intersection of the two lines?

Answer: (1, 2).



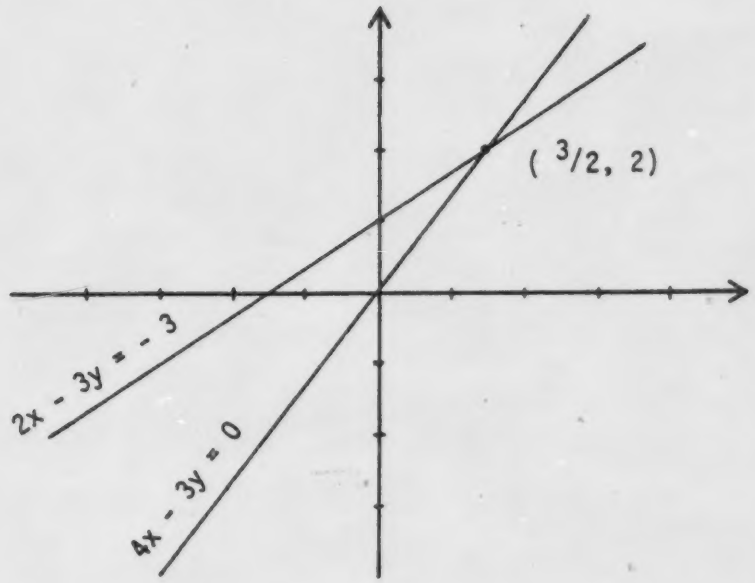
A check of the original equations shows that (1, 2) is a solution for each equation. Thus, using a graphical procedure, we have found the simultaneous solution of the two equations.

Example 2:

Find the solution to the following system of simultaneous equations, using the graphical method.

$$2x - 3y = -3$$

$$4x - 3y = 0 .$$



Exercises:

Use the graphical method to find the point of intersection of the following lines (i.e., the solution to the following systems of simultaneous equations).

1. $x + 2y = 1$
   $2x - y = 3$

2. $\frac{x - y}{2} = 4$
   $5x + y = 1$

3. $x + y = 1$
   $x - y = 1$

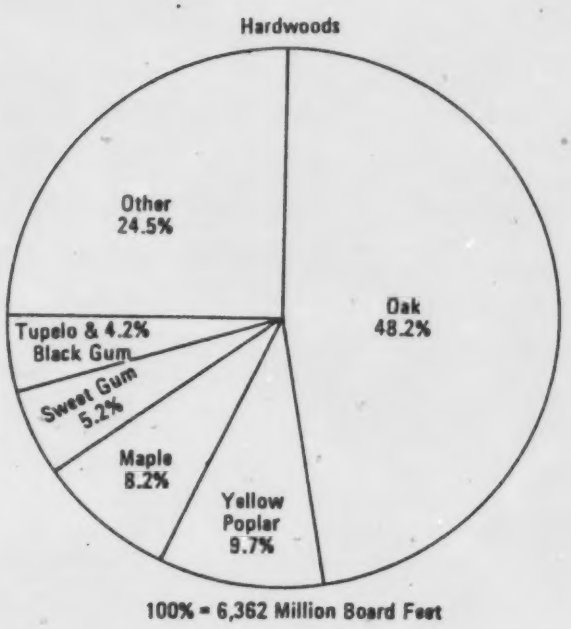
Post-Test

1. The following chart gives the results of a history examination:

Score	53	58	65	69	71	78	82	88	90	98
Frequency	1	3	10	12	14	20	9	7	6	1

Construct a bar graph which represents this data.

2.



The circle graph above shows hardwood production in the U.S. in 1962.

(a) How much oak was produced in 1962?

(b) Black gum, sweet gum, maple and yellow poplar together accounted for what percentage of hardwood production in 1962?

3. The composition of a certain organization is:

neophyte...30
intermediate...45
advanced...25
emeritus...50

Make a circle graph which reflects this breakdown.

4. Plot the following points on a rectangular coordinate system:

$(-5, 1)$, $(3, 1/2)$, $(4, -7)$, $(5/2, -2)$, $(1/4, 3/4)$

5. Write the coordinates of at least 3 points which lie on the graph of $y = 2x + 4$ and sketch the graph.

6. Plot at least 4 points on the graph of $2x^2 + 4y^2 = 9$ and sketch the graph.

7. Use the geographical method to approximate the point of intersection of the following two lines:

$$2x + 3y = 1$$

$$3x - y = 4$$

GEOMETRY AND MEASUREMENT

Introduction

Geometry probably began in ancient Babylon and Egypt as a collection of practical rules for use in surveying and the construction of buildings. The existence of the great pyramids near the Nile is proof that the Egyptians knew their geometry and used it well. This section will introduce you to some fundamental concepts of this subject, including some elementary properties of geometric figures and their measurement.

Objectives

When you have completed this package, you will be able to do the following:

(1) Identify the properties of points, lines, and planes.

(2) Read scales of measuring devices such as rulers, yardsticks and meter sticks.

(3) Measure a line segment sketched on a sheet of paper in English and metric units.

(4) Convert a linear measurement in the English System to a specified larger or smaller unit in the English System.

(5) Convert a linear measurement in the Metric System to a specified larger or smaller unit in the Metric System.

(6) Use a protractor to find the measure of any angle from 0° to 180°.

(7) Use a protractor to construct any angle from 0° to 180°.

(8) Identify the parts of a circle.

(9) Determine the measure of the angles in a triangle, given some properties of the triangle.

(10) Use the Pythagorean theorem to determine whether a given triangle is a right triangle.

(11) Determine the length of the third side of a right triangle, when the lengths of two sides are given.

(12) Determine the length of the diagonal of a square or rectangle, given its dimensions.

(13) Determine the circumference of a circle.

(14) Determine the perimeter of polygons.

(15) Determine the area of circles and polygons.

(16) Determine the area of a circle, given its circumference.

(17) Determine the area (perimeter) of a rectangle, given its perimeter (area) and the length of one side.

Pre-Test

Use Figure T-1 to complete items 1 through 4.

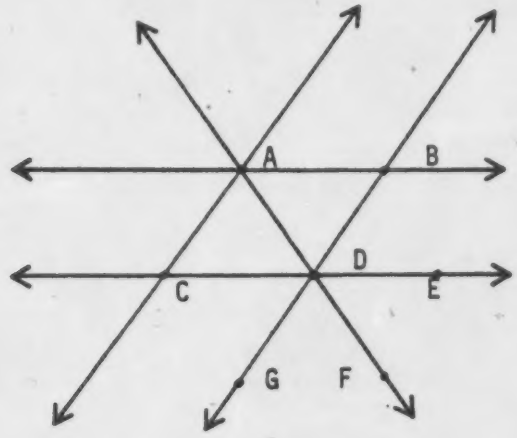


Figure T - 1

1. Name the line segments having E as an end point.

2. Name two pairs of parallel lines.

3. What is the point of intersection for $\overleftrightarrow{AF}$, $\overleftrightarrow{BG}$, and $\overleftrightarrow{CE}$?

4. What plane contains $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$?

5. Use a ruler to measure the following segment in (a) inches and (b) centimeters.

Complete each of the following:

6. 1.2 km = _____m

7. 5 2/3ft. = _____in.

8. 700 cm = _____d

9. 2.9mi. = _____yd.

10. .8 m = _____mm

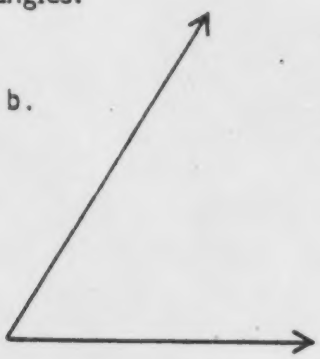
11. 4ft.8in. = _____yd.

12. Name the angles in figure T-1 which have $\overleftrightarrow{DF}$ as a side.

13. Use a protractor to measure the following angles.

a.

b.

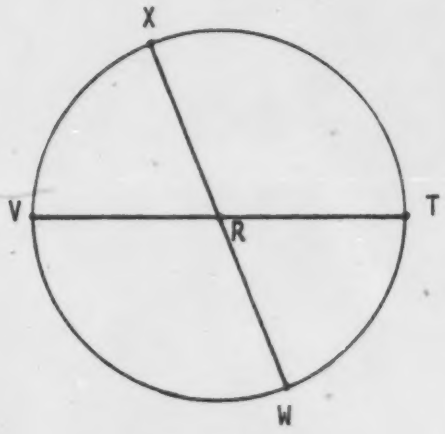


14. Use a protractor to construct the following angles.

a. 137°

b. 79°

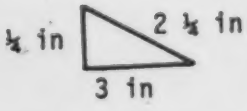
15. Given the circle in figure T-2, complete the following chart.



CENTER	RADII	DIAMETERS	ARC
	RY	ST	$\overset{\frown}{VX}$
			$\overset{\frown}{VW}$

16. In Δ XYZ, $< X = < Y$ and $< Z = 85^{\circ}$. What are the measures of angles X and Y?

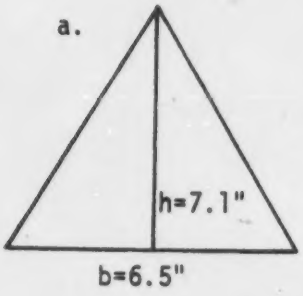
17. Use the Pythagorean theorem to verify that the given triangle is a right triangle.

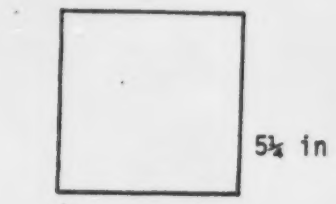


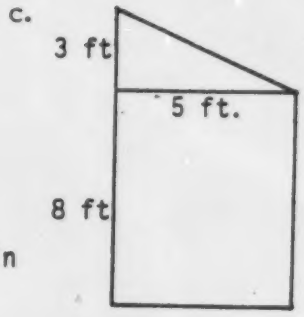
18. A rectangle has dimensions 3 ft. by 4 ft. Find the length of its diagonal.

19. Find the area and circumference of a circle where the radius is 8.5 cm.

20. Find the area and perimeter of each figure below:







21. Find the area of a square where perimeter is 9.2 cm.

Key Terms

point
line
plane
line segment
ray
intersecting lines
skew lines
parallel lines
metric system
meter
kilometer
centimeter
millimeter
angle
vertex
straight angle
degree
protractor
circle
radius
diameter
arc
center
polygon
triangle
isosceles
equilateral
Pythagorean theorem
hypotenuse
quadrilateral
rectangle
square
diagonal
circumference
perimeter
area

Study Materials

Points, Lines, and Planes

A mathematical development typically begins with a mutual agreement on the meanings of certain fundamental terms. Such terms are said to be undefined. The most fundamental concepts of geometry are Point, Line, and Plane. However, each has certain properties such that a physical representation can be made.

A representation for a point can be made if you press the sharp end of a pencil on a piece of paper. You will make a dot. The dot is a physical representation for a point. However, the dot has measurable dimensions while a point does not. The point is not the dot, but rather the position it occupies.

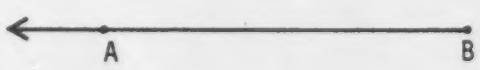
.P (point P)

A point is usually indicated by a dot named by a capital letter placed near it.

A line may be thought of as an infinite set of points which extends indefinitely in both directions.

(line l or $\overleftrightarrow{AB}$)

A half-line or RAY extends indefinitely in only one direction.



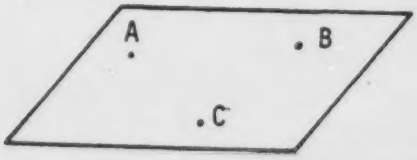
(ray $\overleftrightarrow{AB}$)

(ray $\overleftrightarrow{AB}$)

A line of definite length is called a Line Segment.

(Line Segment $\overleftrightarrow{AB}$)

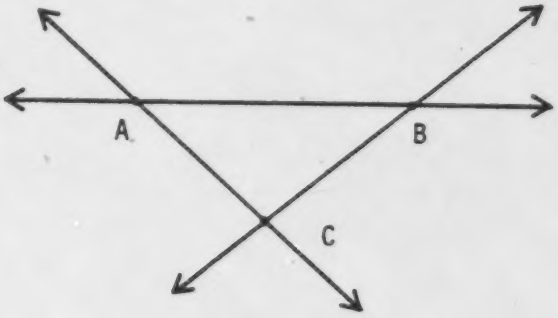
A plane may be thought of as a flat surface which extends indefinitely in all directions.



(Plane ABC)

Approximations of planes include a sheet of paper, a pane of glass, the walls, ceiling, and floor of a room, etc.

To further clarify the concept of a plane, consider the following. Suppose we have three points not all on the same line. Then each pair of these points determines a line.



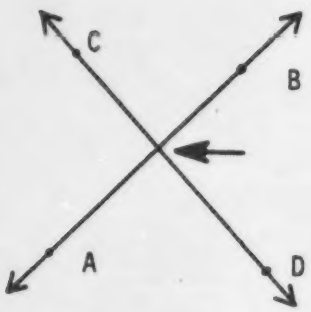
In addition, each pair of distinct points on lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$ determines a line, and each pair of points on these lines determines a line, and so on. The totality of all points on all of these lines constitutes the plane determined by A, B, and C.

Several fundamental assumptions can be made about points, lines, and planes. They are:

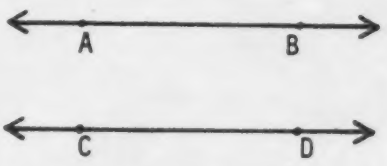
1. Through two distinct points there is exactly one straight line.

2. Through three distinct points not on the same line there is exactly one plane.

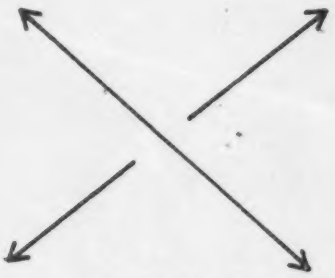
3. Two distinct straight lines have in common either one point or no point at all. If they have a point in common, they are said to INTERSECT. If they lie in a plane and do not intersect, they are called PARALLEL. Two lines which do not intersect and which do not lie in the same plane are called SKEW.



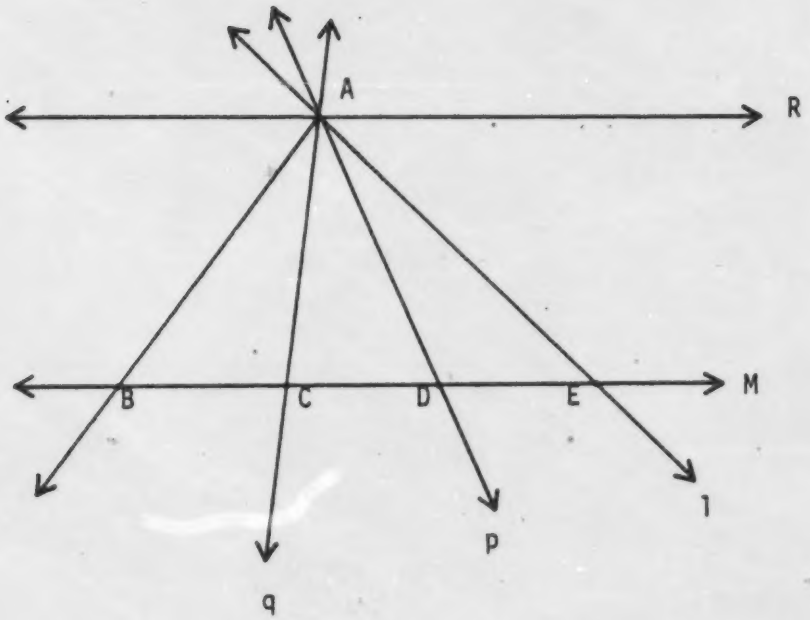
Intersecting Lines



Parallel Lines

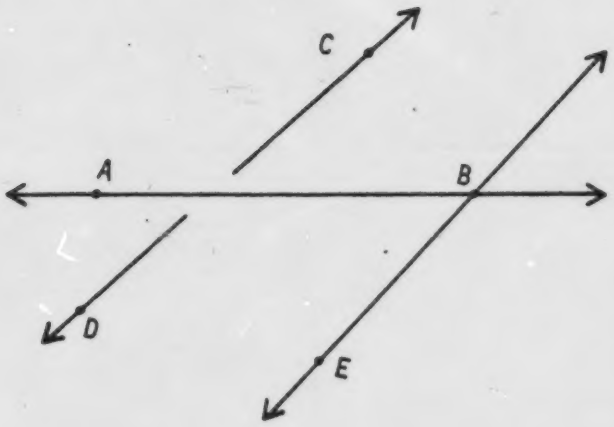


Skew Lines



Use the figure above to answer the following questions.

1. Which points are on line m?
2. Which points are on line l?
3. Which pairs of lines intersect?
4. Which pairs of lines are parallel?
5. Which ray has A as an end point?
6. Which line segments have E as an end point?



Use the figure above to determine whether the following statements are true or false.

7. $\overleftrightarrow{BE}$ lies in the plane determined by points A, B, and E.

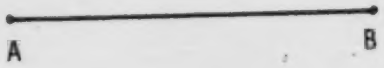
8. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ do not intersect and are therefore parallel.

Measurement of Length

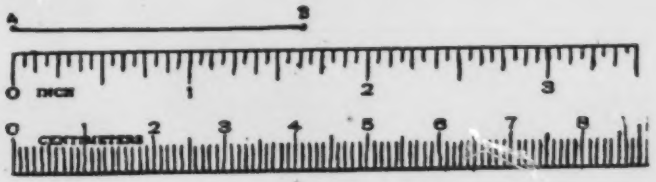
Length is an important characteristic of a line. The part of a line whose length is to be determined is a line segment. The most familiar instruments for measuring length are the twelve-inch ruler, the yardstick, and the meter stick. Each measuring device is divided into given units and fractions of the unit. For example, consider the ruler below. The upper edge is divided into inches and fractional parts of an inch (sixteenths).

The lower edge is divided into centimeters and fractional parts of a centimeter (tenths).

Example: Measure the length of the line segment below.



In order to measure the length of this line segment, place the division marked zero on the ruler at A, with the edge of the ruler along side $\overleftrightarrow{AB}$. Next, find the reading on the ruler which corresponds to point B.



This reading is 1 10/16 or 1 5/8 inches. Note that on the centimeter scale, this is approximately 4 1/10 centimeters. The standard units for measuring length in the English System are the inch, the feet, the yard, and the mile. The relationships between these units are given below.

UNITS OF LINEAR MEASURE		
12 inches (in.)	=	1 foot (ft.)
3 feet	=	1 yard (yd.)
1,760 yards	=	1 mile (mi.)
5,280 feet	=	1 mile (mi.)

The standard metric measure of length is the meter. Prefixes which increase or decrease in powers of 10 are used with this unit to provide larger or smaller units for different measurement situations.

Prefix	Definition
milli	1/1000 of
centi	1/100 of
deci	1/10 of
deca	10 times
hecto	100 times
kilo	1000 times

METRIC MEASURES OF LENGTH		
10 millimeters	=	1 centimeter (cm)
10 centimeters	=	1 decimeter (dm)
10 decimeters	=	1 meter (m)
10 meters	=	decameter (dam)
10 decameters	=	1 hectometer (hm)
10 hectometer	=	1 kilometer (km)

Observe that:

1 kilometer is 1000 meters

1 centimeter is 1/100 of a meter or 100 centimeters = 1 meter

1 millimeter is 1/1000 of a meter or 1000 millimeters = 1 meter

Some useful metric referents are:

1 mm is about the thickness of a dime.

5 mm is about the length of an eraser on a new pencil.

1 cm is about the width of a popsicle stick.

2 cm is about the diameter of a nickel.

1 m is about the height of a five-year-old child.

2 m is about the height of a door in a house.

Conversions

Sometimes it is necessary to change a given unit of measurement to a larger or smaller unit of measurement. This can be accomplished by proceeding as follows:

1. To change a given unit of measurement to a larger unit of measurement, divide the given number of units by the number of times the smaller unit is contained in the larger unit.

2. To change a given unit of measurement to a smaller unit of measurement, multiply the given number of units by the number of times the smaller unit is contained in the larger unit.

Example 1: Change 38 inches to feet.

Solution: We are changing a smaller unit to a larger unit of measurement. Thus, we must divide 38 by the number of inches contained in one foot (12).

$$38 \div 12 = 3\frac{2}{12} = 3\frac{1}{6} \text{ feet.}$$

Therefore, 38 inches = 3 1/6 feet.

Example 2: Change 60 centimeters to meters.

Solution: We are changing a smaller unit to a larger unit of measurement. Thus, we divide 60 by the number of centimeters contained in one meter (100).

$$60 \div 100 = .6$$

Therefore, 60 centimeters = .6 meter

Example 3: Change 2 3/4 miles to yards.

Solution: We are changing a larger unit to a smaller unit of measurement. Thus, we must multiply 2 3/4 by the number of yards contained in 1 mile (1,760).

$$2\frac{3}{4} \times 1760 = \frac{11}{4} \times 1760 = 4840$$

Therefore, 2 3/4 miles = 4,840 yards.

Example 4: Change 1.92 kilometers to millimeters.

Solution: We are changing from a larger unit to a smaller unit of measurement. Thus, we must multiply 13.8 kilometers by the number of millimeters contained in one kilometer.

1,000 millimeters = 1 meter.

1 kilometer = 1,000 (1,000 millimeters)
Hence, 1 km = 1,000,000 millimeters

1.92 x 1,000,000 = 1,920,000

Therefore, 1.92 km = 1,920,000 millimeters

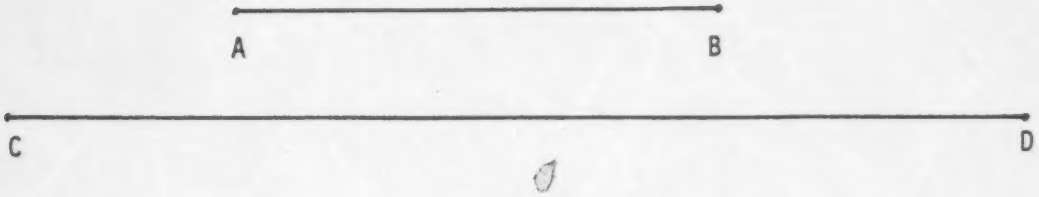
Example 5: Change 8 ft. 4 in. to feet.

Solution: 4 inches = 4/12 ft. = 1/3 ft.

Therefore, 8 ft. 4 in. = 8 1/3 ft.

Practice Problems

1. With a ruler, measure $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ in (a) inches and (b) centimeters.



2. Estimate your height in meters and then measure your height with a meter stick. Compare the two measurements.

3. Repeat problem (2) with a yardstick.

Change each of the following to the units indicated.

4. 42 in. = _____ ft.

5. $5\frac{1}{3}$ yd. = _____ ft.

6. 3 yd. 2 ft. = _____ in.

7. 9240 ft. = _____ mi

8. 600 cm = _____ m

9. 5.8 m = _____ mm

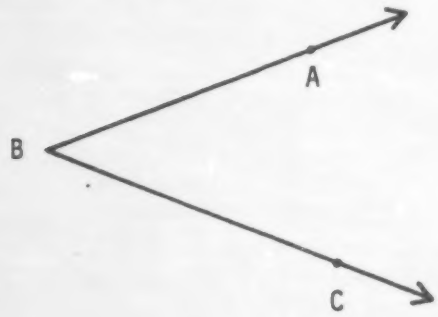
10. 3,486,000 mm = _____ km

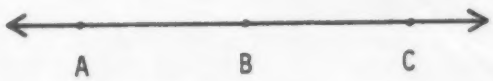
11. 2.9 km = _____ m

12. .15 hm = _____ dam

Angular Measurement

An ANGLE is a geometric figure consisting of two distinct rays with a common endpoint. The rays are the sides of the angle and the common endpoint is the VERTEX of the angle. If the rays are opposite rays, the angle is called a STRAIGHT ANGLE.



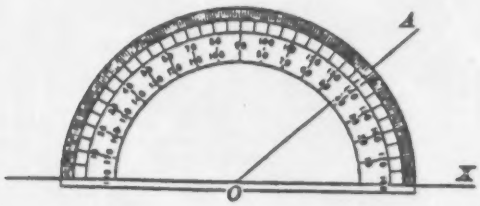


Straight Angle

These angles are denoted by < ABC or < _. The symbol " < " is read "angle". When three letters are used to denote an angle, the vertex is the middle one of the three letters. When one letter is used to denote an angle, it must be the letter corresponding to the vertex.

In order to measure an angle, we need a unit of measurement which is itself an angle. If a straight angle is divided into 180 equal parts, the unit of measurement used to measure angles is obtained. This unit is called a DEGREE. One degree (1°) is 1/180 of a straight angle.

The instrument used to measure given angles is a PROTRACTOR. For example, to measure < AOX below, place the protractor so that its center falls on vertex 0 of the angle. The lower edge of the protractor should lie along side $\overline{OA}$ of the angle. Where side $\overline{OX}$ of the angle

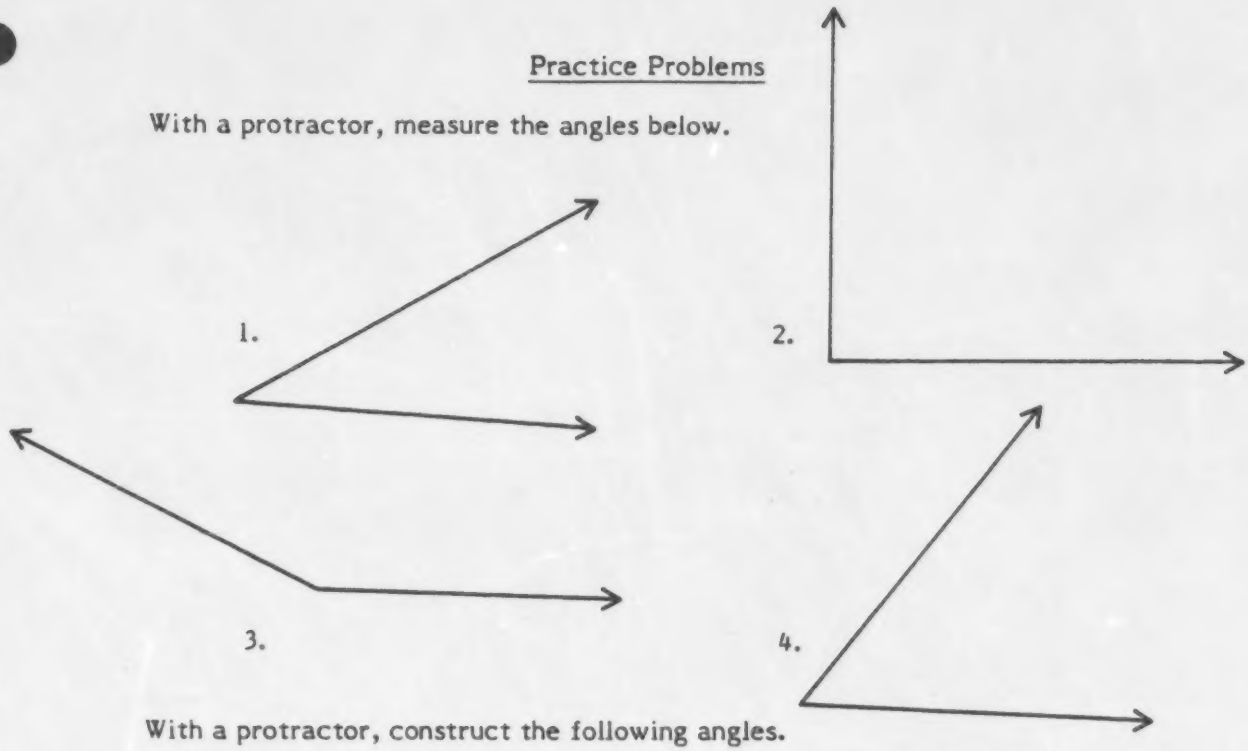


intersects the outer edge of the scale on the protractor, read the number of degrees in < AOX. Here, < AOX = 42°.

The protractor can also be used to construct angles of a desired size. Suppose, for example, you wished to construct an angle of 42° (say AOX). First, you must draw a baseline of which $\overline{OA}$ is a part. On this line mark point 0 as the vertex of the angle to be drawn. Place the protractor with its center at O and its lower edge on $\overline{OA}$. Next, place a ruler so that it passes through O and the 42° mark on the protractor scale. Mark a point X along the ruler, beyond the outer edge of the protractor. Remove the protractor and draw line $\overline{OX}$.

Practice Problems

With a protractor, measure the angles below.



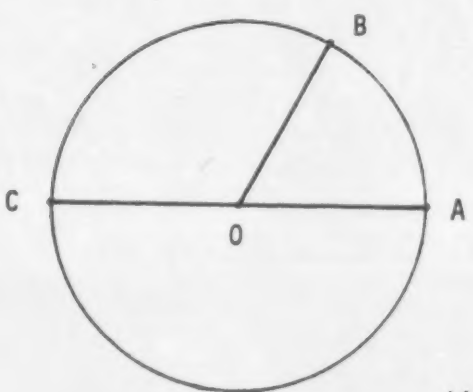
With a protractor, construct the following angles.

5. 20°
6. 45°
7. 72°
8. 167°

Plane Figures

Circles

A CIRCLE is a plane figure consisting of a set of points which are a fixed distance from a given point. The given point is the CENTER, and the given distance is called the RADIUS.

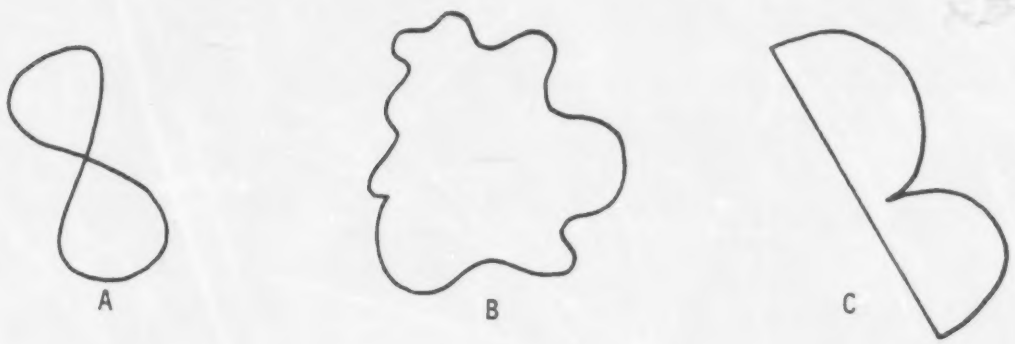


0 is the center of this circle. Any line segment from the center to a point on the circle is called a radius of the circle. $\overline{OA}$ is a radius of the given circle.

The DIAMETER of the circle is $\overline{AC}$. The diameter of a circle is any line segment which passes through the center of the circle and whose end points lie on the circle. The diameter is also twice the length of the radius. $\overset{\frown}{AB}$ (read arc AB) is the part of the circle between points A and B.

If you start at the point A on the circle above and trace the curve, you will eventually return to point A. Any curve having this property is said to be closed. In addition, a closed curve that does not cross itself is called a simple closed curve. It is easy to see that a circle is a simple closed curve.

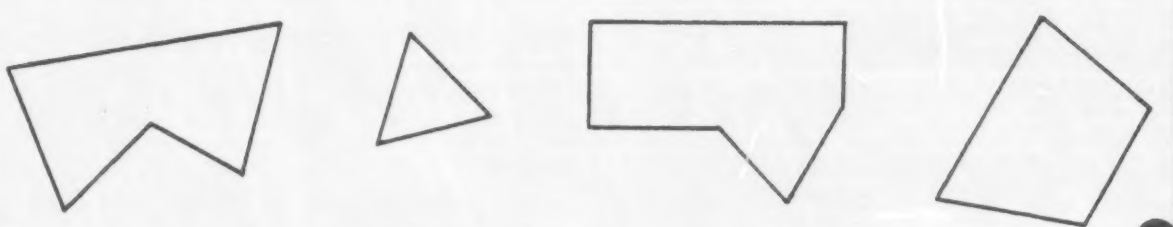
To further clarify this idea, consider the closed curves below.



Only one of these closed curves (A) is not simple.

Polygons

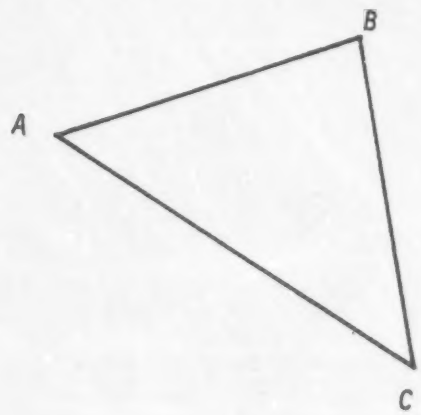
A polygon is a simple closed curve formed by line segments drawn end to end. (In geometry a line segment is a special case of a curve.) Each of the following figures is a polygon.



Polygons are classified according to the number of sides they have.

NUMBER OF SIDES	NAME
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon

The most common polygons are triangles and quadrilaterals. The triangle is the polygon with the smallest number of sides. Consider the triangle below.



We refer to this as Δ ABC (read triangle ABC). Points A, B, and C are the vertices of the triangle. (Each is the vertex of an angle.) The three angles are $\angle A$, $\angle B$, and $\angle C$. The three sides of the triangle are line segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. A triangle is ISOSCELES if two of its sides have the same length. It is EQUILATERAL if all three sides have the same length. It is a property of triangles that the sum of their angles is 180°. In an Isosceles triangle, two of the angles will have the same measure. In an equilateral triangle, all angles have the same measure.

$\overline{AB} = \overline{BC} = \overline{AC}$

$< A = < B = < C$

Equilateral Triangle

$\overline{DE} = \overline{DF}$

$< D = < F$

Isosceles Triangle

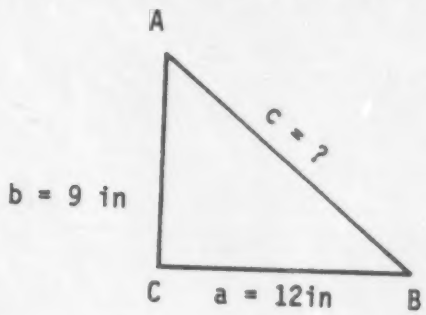
If one angle of a triangle is a right angle (90°), the triangle is called a right triangle. A famous relation between sides a, b, and c of a right triangle is

$$c^2 = a^2 + b^2,$$

where c is the HYPOTENUSE (the length of the side opposite the right angle).

This relation is called the Pythagorean theorem.

Example 6: In right triangle ABC, find the length of side c.



Solution: Using the Pythagorean theorem, we have

$$c^2 = a^2 + b^2$$

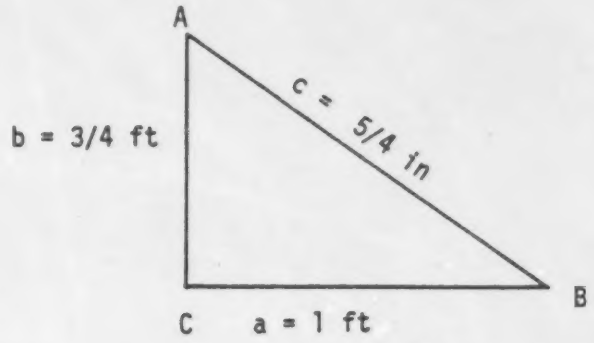
$$c^2 = 12^2 + 9^2$$

$$c^2 = 144 + 81$$

$$c^2 = 225 = (15)^2$$

Therefore, C = 15 in.

Example: Verify that Δ ABC is a right triangle.



Solution: Δ ABC is a right triangle if $\frac{5}{4}^2 = 1^2 + \frac{3}{4}^2$.

$$\frac{5}{4}^2 \stackrel{?}{=} 1^2 + \frac{3}{4}^2$$

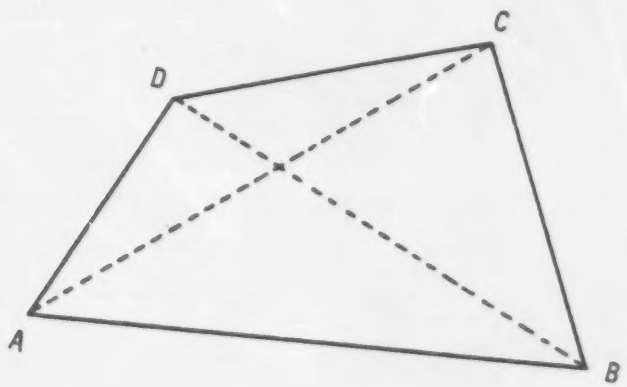
$$\frac{25}{16} \stackrel{?}{=} 1 + \frac{9}{16}$$

$$\frac{25}{16} \stackrel{?}{=} \frac{16}{16} + \frac{9}{16}$$

$$\frac{25}{16} = \frac{25}{16}$$

Since $c^2 = a^2 + b^2$, Δ ABC is a right triangle.

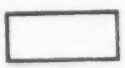
Quadrilaterals are polygons with four sides. Consider the quadrilateral below.



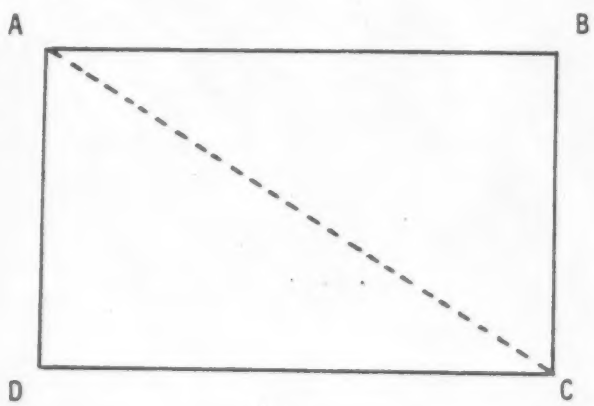
The points A, B, C, and D are called the vertices of the quadrilateral. Segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$ are called the sides of the quadrilateral. $\overline{BD}$ and $\overline{AC}$ are called DIAGONALS of the quadrilateral. Diagonals are segments which join opposite pairs of vertices.

It is a property of quadrilaterals that the sum of their angles is 360°.

Certain quadrilaterals are given special names according to their characteristics.

NAME	FIGURE	CHARACTERISTICS
Parallelogram		2 pairs of parallel sides
Rectangle		2 pairs of parallel sides, 4 right angles
Square		2 pairs of parallel sides, 4 right angles, 4 sides of equal length
Trapezoid		1 pair of parallel sides

Example 8: Find the length of diagonal d in the rectangle below.



Solution: Δ ACD is a right triangle. Thus, we may use the Pythagorean theorem to determine d.

$$d^2 = a^2 + c^2$$

$$d^2 = 8^2 + 6^2$$

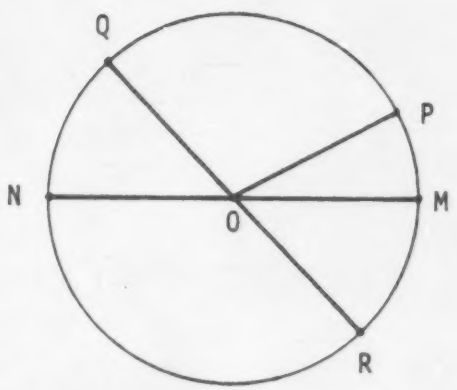
$$d^2 = 64 + 36$$

$$d^2 = 100 = (10)^2$$

$$d = 10\text{in.}$$

Practice Problems

1. Complete the chart below for the given circle.



CENTER	RADII	DIAMETERS	ARCS
	$\overline{OP}$		$\overset{\frown}{PM}$

2. Draw a triangle. Use a protractor to determine the measure of each angle. Compute their sum.

3. In triangle PQR, P = 32° and Q = 119°.

   What is < R?

4. In triangle ABC, $\angle A = \angle B$ and $\angle C = 80^\circ$.

   What are $\angle A$ and $\angle B$?

5. In triangle EFG, $\overline{EF} = \overline{FG} = \overline{GE}$

   What are $\angle E$, $\angle F$, and $\angle G$?

6. Construct a right triangle, making the sides, including the right angle, 3 and 4 units long, respectively. Using the same unit, find the length of the hypotenuse. How does the square of the hypotenuse compare with the sum of the squares of the other two sides? Use Pythagorean theorem to verify your result.

Measurement of Plane Figures

Circumference

The CIRCUMFERENCE of a circle is the length ot the closed curved line which forms the circle. The ratio of the circumference (c) to the diameter (d) is a number which is the same for all circles. The Greek letter π (pi, pronounced "pie") is used to represent this ratio. The value of π is approximately 3.14 or 22/7. Thus,

$$\frac{c}{d} = \pi$$

The circumference of a circle is π times the diameter or π times twice the radius. That is,

$$c = \pi d \qquad \text{or} \qquad c = 2 \pi r$$

Example 9: Find the circumference of a circle whose diameter is 6 inches.

Solution:

$$c = \pi d$$

$$= (3.14)(6 \text{ inches})$$

$$= 18.84 \text{ inches}$$

Example 10: Find the circumference of the circle whose radius is 7/8 ft.

Solution:

$$c = 2 \pi r$$

$$= 2 (22/7) (7/8 \text{ ft})$$

$$= 44/8 \text{ ft.}$$

$$= 5\tfrac{1}{2} \text{ ft.}$$

Example 11: Find the diameter of a circle whose circumference is 20 inches.

Solution:

$$d = \frac{c}{\pi}$$

$$d = \frac{20 \text{ inches}}{3.14}$$

$$d = 6.37 \text{ inches}$$

Practice Problems

Find the circumference of the circles with the following diameters.

(Use $\pi = 3.14$)

1. 6 ft.
2. 18 yds.
3. 25 cm

Find the circumference of the circles with the following radii.

(Use $\pi = 22/7$)

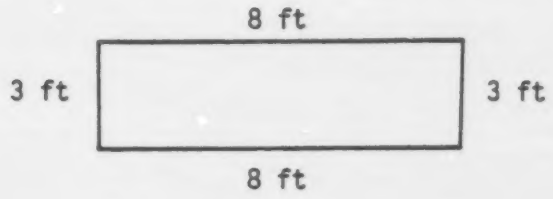
4. $3\frac{1}{2}$ in.
5. $\frac{1}{4}$ m
6. 70 cm

7. What is the radius of a circle whose circumference is 18 feet?

Perimeter

The distance around a polygon, or the sum of the lengths of all sides, is called the PERIMETER of the polygon.

Example 12: Find the perimeter (P) of the rectangle below.

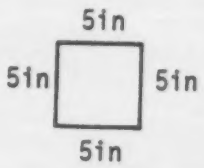


Solution:

$$P = 8\text{ ft} + 3\text{ ft} + 8\text{ ft} + 3\text{ ft}$$
$$= (8\text{ ft} + 8\text{ ft}) + (3\text{ ft} + 3\text{ ft})$$
$$= 2(8\text{ ft}) + 2(3\text{ ft})$$
$$= 16\text{ ft} + 6\text{ ft}$$
$$= 22\text{ ft}$$

Note: The perimeter of a rectangle can be found by using the formula $P = 2l = 2w$, where l represents the length and w is the width.

Example 13: Find the perimeter of the square below.



Solution:

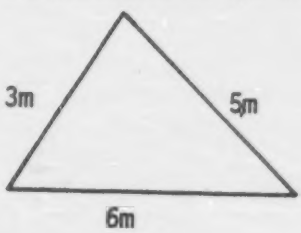
$P = 5 \text{ in} + 5 \text{ in} + 5 \text{ in} + 5 \text{ in}$

$= 4 (5 \text{ in})$

$= 20 \text{ in.}$

The perimeter of a square can be found by using the formula $P = 4S$, where S represents the length of a side.

Example 14: Find the perimeter of the triangle below.



Solution:

$P = 3 \text{ m} + 5 \text{ m} + 6 \text{ m}$

$= 14 \text{ m}$

The perimeter of a triangle may be found by using the formula
$P = a + b + c$,
where a, b, and c represent the lengths of sides

Practice Problems

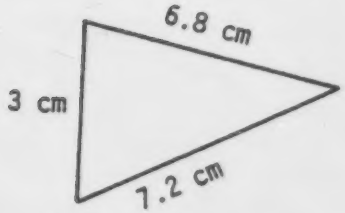
1. Find the perimeter of a square for which one side measures:

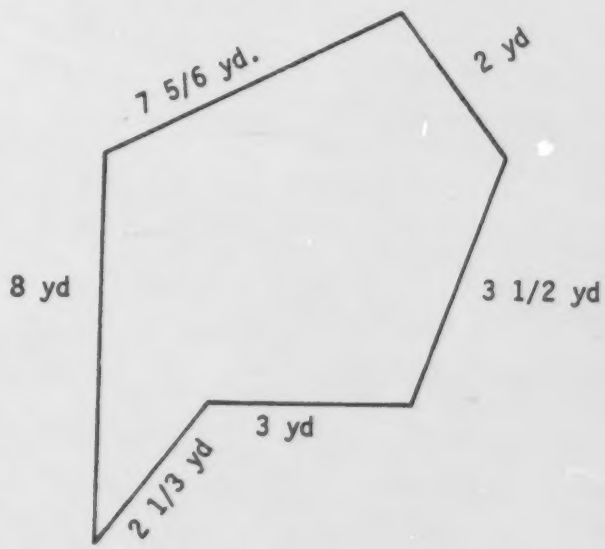
   a. 4.2 m b. $2\frac{5}{6}$ yd.

2. Find the perimeter of the rectangle whose length (l) and width (w) are:

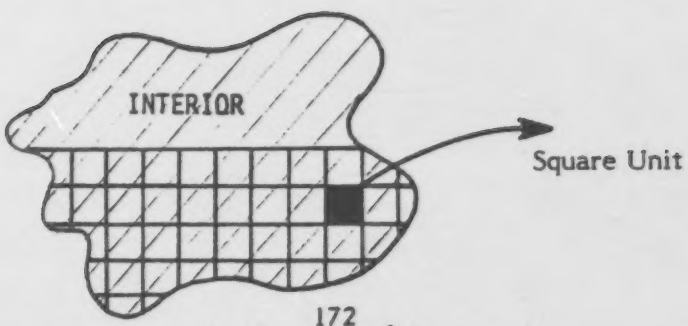
   a. $l = 5$ ft, $w = 10$ ft. b. $l = 3.3$ cm, $w = 5$ cm

3. Find the perimeter of each polygon below:





Area



The measure of the interior of a simple closed curve is called AREA. The unit of area is a square each of whose sides is a standard unit of length. Such a unit involves length and width. Thus, we may measure area and express the result in square feet, square inches, square meters, square centimeters, etc.

The formulas for finding the area of some familiar geometric figures are given below.

FIGURE	AREA (A)
Circle	$A = r^2$, r is the radius
Rectangle	$A = lw$, l is the length, w is the width
Square	$A = S^2$, S is the length of a side
Triangle	$A = \frac{1}{2}bh$, b is the base of the triangle, h is its height

Example 15: Find the area of a circle whose radius is 4 in.

Solution:

$$A = \pi r^2$$

$$= (3.14)(4 \text{ in})^2$$

$$= (3.14)(16 \text{ sq in.})$$

$$A = 50.24 \text{ sq. in.}$$

Example 16: Find the area of a circle whose circumference is 31.4m.

Solution: $c = 2\pi r$, lc

$$R = \frac{C}{2\pi}$$

$$R = \frac{31.4m}{2(3.14)}$$

$$R = 5m$$

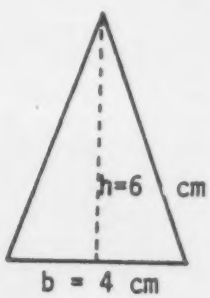
$$A = \pi r^2$$

$$= (3.14)(5m)^2$$

$$= (3.14)(25 \text{ sq.m})$$

$$= 78.5 \text{ sq.m.}$$

Example 17: Find the area of the triangle below.



Solution: $A = \frac{1}{2}bh$

$$= \frac{1}{2}(4cm)(6cm)$$

$$= \frac{1}{2}(24 \text{ sq. cm})$$

$$= 12 \text{ sq. cm.}$$

Example 18: Find the area of the rectangle whose length is 2 ft and whose width is 5 1/2 ft.

Solution: $A = lw$

$A = (2 \text{ ft})(5\frac{1}{2} \text{ ft})$

$A = (2 \text{ ft})(\frac{11}{2} \text{ ft})$

$A = 11 \text{ sq. ft.}$

Example 19: Find the area of a square if its perimeter is 28 mm.

Solution: $P = 4S$, SO,

$$S = \frac{P}{4}$$

$$S = \frac{28 \text{ mm}}{4}$$

$$S = 7 \text{ mm}$$

$A = S^2$

$A = (7\text{mm})^2$

$A = 49 \text{ sq. mm.}$

Example 20: Find the area of a rectangle whose perimeter is 22 ft. and whose length is 5 ft.

Solution:

$P = 2(l) + 2(w)$,SO

$22 = 2(5) + 2w$

$22 = 10 + 2w$

$12 = 2w$

$6 = w$

Since $w = 6$ ft , and $l = 5$ ft.

$A = l\,w$

$= (5 \text{ ft})(6 \text{ ft})$

$= 30$ sq. ft.

Practice Problems

Find the area of the circles whose radii are given.

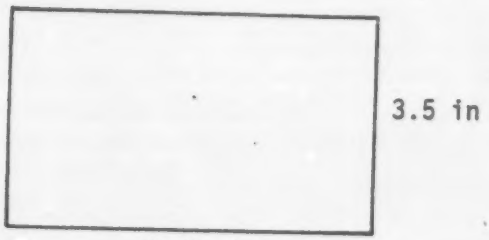
1. R = 4.1 cm

2. R = 10 m

3. Find the area of the circle whose circumference is 25.12 ft.

Find the area of the figures given.

4.



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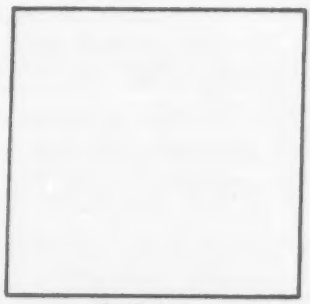
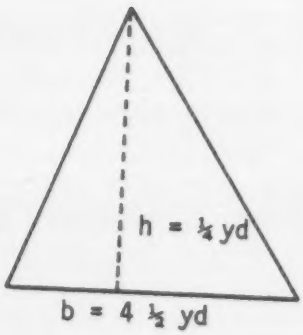
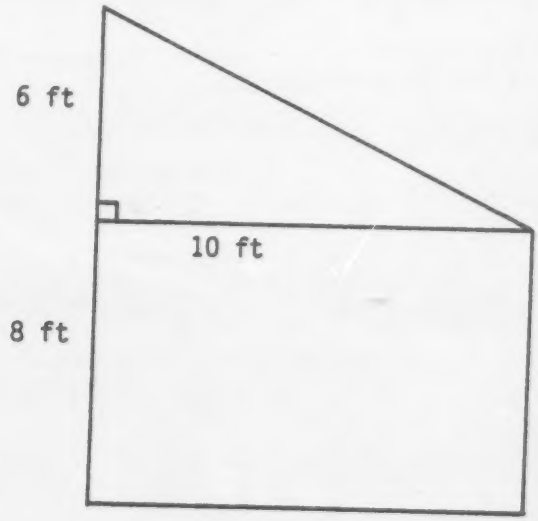


6.



7.



8. Find the area of a rectangle whose perimeter is 42 in. and whose width is 6 in.

Post-Test

Use figure T-1 to complete items 1 through 5.

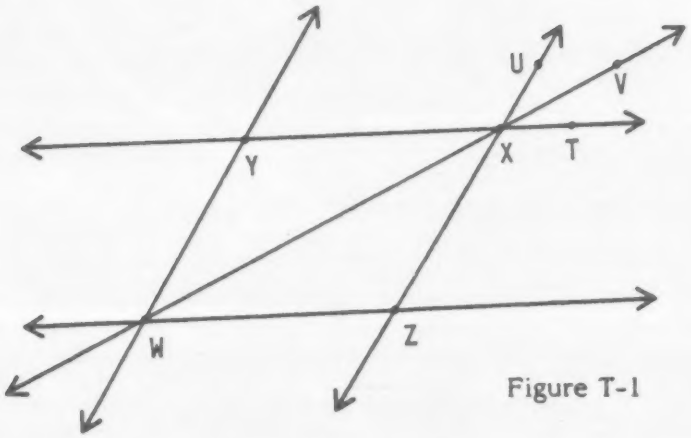


Figure T-1

1. Name two lines segments having T as an end point.

2. Name two pairs of parallel lines.

3. What is the point of intersection for $\overleftrightarrow{VW}$, $\overleftrightarrow{UZ}$, and $\overleftrightarrow{TY}$?

4. What plane contains $\overleftrightarrow{YW}$ and $\overleftrightarrow{WZ}$?

5. Use a ruler to measure the following segment in (a) inches and (b) centimeters.

Complete each of the following.

6. .018 km = _____ dm

7. $4\frac{3}{4}$ ft. = _____ yd.

8. 624 cm = _____ m

9. 8800 yd = _____ mi

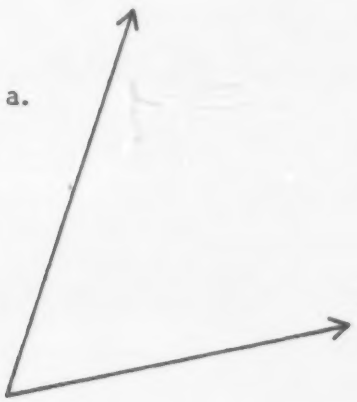
10. 17 hm = _____ mm

11. 3 yd. 10 in = _____ ft.

12. Name the angles in figure T-1 which have TX as a side.

13. Use a protractor to measure the following angles:

a.



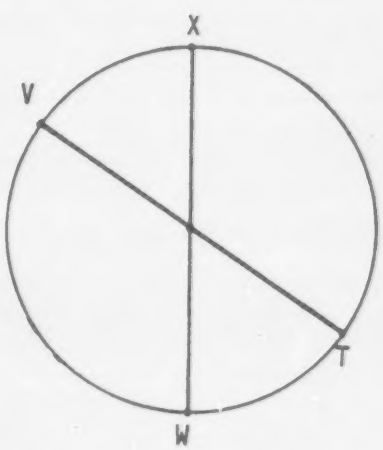
b.

14. Use a protractor to construct the following angles:

a. 28° b. 109°

15. Given the circle in figure T-2, complete the following chart.

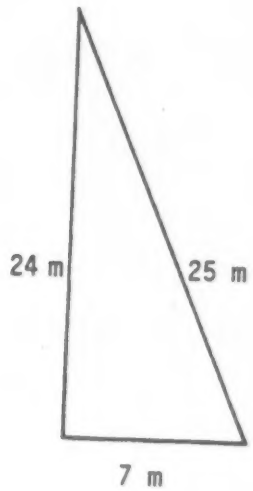
CENTER	RADII	DIAMETERS	ARCS



16. In Δ ABC, $< A = 10$ and $< B = 92^{o}$.

What is $< C$?

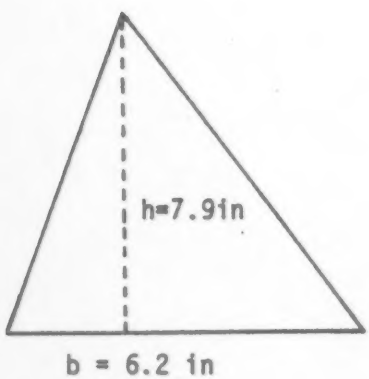
17. Use the Pythagorean theorem to verify that the given triangle is a right triangle.

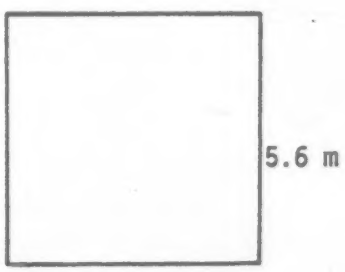


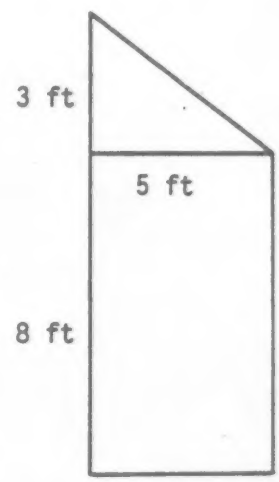
18. What is the length of the diagonal of a square that measures 4 in. on one side?

19. Find the area and circumference of a circle whose radius is 2.5 ft.

20. Find the area and perimeter of each figure below:







21. Find the area of a rectangle whose perimeter is 20m and whose length is 6m.

BASIC STATISTICS

Introduction

This section introduces you to Descriptive Statistics, which has in recent years pervaded all disciplines. It is becoming increasingly evident that a basic knowledge of statistics is important for all people who need to analyze voluminous amounts of data and report their findings in an economical method. Lacking a knowledge of statistics, one is vulnerable to deceptive tactics employed by many salespersons or lecturers. It is herein assumed that educated people should be capable of detecting statistical lies, and thus be able to use appropriate statistical reasoning.

The only pre-requisite for this learning section is a mastery of basic computational skills.

Objectives

At the end of this section you should be able to:

(1) Write the mean, median, mode, and midrange without error, given a 10-item set of numerical data.

(2) Construct an ungrouped frequency table without error, given a 20-item set of numerical data.

(3) Write the percentile rank of each item, given a set of no more than twenty data items.

(4) Construct a frequency polygon of grouped data without error, given a 20-item set of numerical data.

(5) Construct a frequency table of the data in grouped form without error, given a 20-item set of numerical data and a class interval width.

(6) Compute the mean of the squared deviations (variance) to two decimal places, given a 10 item set of numerical data.

(7) Compute the standard deviation to one decimal place, given a 10-item set of numerical data.

Pre-Test

Ten students were asked how many children the mother of each had. Use the data below to answer questions 1 through 6.

Number of Children: {9, 6, 5, 4, 6, 2, 6, 3, 7, 2}

1. The mean number of children is

   (a) 50 (b) 5.0 (c) 10 (d) 5.5

2. The median for the number of children is

   (a) 5.0 (b) 5.5 (c) 10 (d) 4

3. The mode for this set of data is

   (a) 5 (b) 2 (c) 6 (d) No mode

4. The midrange for these survey results is

   (a) 6 (b) 5.0 (c) 5.5 (d) 7

5. The standard deviation to the nearest tenth is

   (a) 2.3 (b) 2.1 (c) 4.6 (d) 10

6. The variance to two decimal places is

   (a) 0.00 (b) 46.00 (c) 4.60 (d) .46

The following set of test data should be used to answer questions 7 through 10.

60, 60, 61, 70, 70, 72, 72, 72, 73, 74,

74, 74, 80, 81, 81, 81, 85, 85, 90, 90

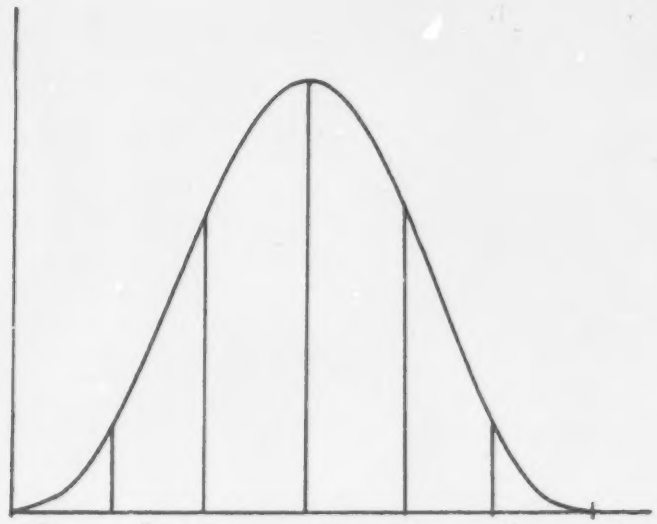
7. Construct a frequency table of the data.

8. Construct a frequency table of the data grouped in intervals of length 3.

9. Construct a frequency polygon of the data as grouped in problem 8.

10. Find the percentile rank of a raw score of 74.

    (a) 55 (b) 45 (c) 50 (d) 60



Use Figure 1 for problems 11 and 12.

11. Find the percentile rank for a score of 74 from a normally distributed set of data with mean 62 and standard deviation 6.

(a) 2.5 (b) 97.7 (c) 84.1 (d) 74

12. Find the percentile rank for a score of 40 from a normally distributed set of data with mean 47 and standard deviation 7.

(a) 33.3 (b) 50 (c) 40.0 (d) 15.9

Key Terms

central tendency
mode
midrange
frequency table
class interval
dispersion
variance (σ^2)
normal distribution
mean ($\bar{X}$)
median
frequency distribution
frequency polygon
percentile
range
standard deviation (σ)
normal curve

Study Materials

Measures of Central Tendency

What are some characteristics of people which may be designated by numbers?

List at least three such characteristics as headings for the table below. Complete the chart by filling in the rows with numerical data as it corresponds to each student in the class. After this is done, answer the questions below.

TABLE I

1.			
2.			
3.			
4.			
5.			
6.			
7.			
8.			
9.			
10.			
11.			
12.			
13.			
14.			
15.			
16.			
17.			
18.			
19.			
20.			
21.			
22.			
23.			
24.			
25.			

Questions:

1. For each column of data in your chart, compute the "average".

2. Examine the data in each column carefully. Which data item in each column appears most often?

3. Re-order the data in each column from smallest to largest. If there are data items that repeat, write them as often as they appear.

   Which data item appears in the middle of your list for each set of data?

   Which number would you use for the middle if you had an even number of data items?

4. Determine the smallest item in each list and the greatest item. Find the "average" of these.

The results that you have found in questions 1 through 4 are called measures of central tendency of a data set. These are numbers about which the data clusters.

Definitions

The mean ($\bar{X}$) of a set of numerical data is the sum of the data items divided by the number of data items.

The median of a set of numerical data is the middle number of the ranked set of data items.

The mode of a set of numerical data is the data item that appears most frequently.

The midrange of a set of numerical data is the arithmetical average of the smallest and greatest data items.

Frequency Tables

Ungrouped Data

The number of items in the sets of data that you have been working with is small in comparison to the number of items in many real life situations. With a list of data items such as the 50 mathematics placement test scores below, oftimes a frequency table is used to order data and represent it in summary form.

MATHEMATICS PLACEMENT TEST SCORES (%ile)

81, 82, 85, 97, 98, 96, 95, 81, 77, 78,

78, 72, 87, 88, 91, 95, 82, 80, 73, 84,

75, 82, 81, 90, 93, 92, 94, 78, 72, 73,

90, 91, 93, 92, 91, 95, 90, 85, 82, 81,

90, 93, 94, 72, 78, 79, 80, 81, 83, 82

Use the frequency table below, containing tally and frequency columns for each score, to record the data above. The first 10 horizontal scores have already been recorded. Follow the pattern and complete the table.

TABLE II

Mathematics Placement Test Scores (%ile)

Score	Tally	Frequency	Score	Tally	Frequency
70			85		
71			86		
72			87		
73			88		
74			89		
75			90		
76			91		
77			92		
78			93		
79			94		
80			95		
81			96		
82			97		
83			98		
84			99		

If you filled in your table correctly, it should look like the one in the Appendix at the end of this section. Check your results. If you disagree with the table in the Appendix, ask your teacher or laboratory assistant to help you to make necessary corrections.

Exercise:

Compute the measures of central tendency for the data above using the frequency table that you have just completed.

Percentiles of Raw Scores

1. Select a given score in the frequency table. _____
2. Sum the frequencies of all scores less than the given score. _____
3. This sum is what percent of the total number of frequencies? _____
4. The result is the percentile rank of the score. _____

Example:

Given the score 92 from Table II, the sum of the frequencies of the scores that are less than 92 is 37. Since there are 50 scores and 37 is 74 percent of 50, the percentile rank of 92 is 74. Note that one can apply the same rules for a score with a frequency of 0 and determine its percentile rank. How do you think this rank compares with percentile rank of the next larger score with a non-zero frequency? Convince a classmate that you are correct.

Grouped Data

There are many occasions when data is reported in a condensed or grouped form, and in these instances an interval size must be determined, as well as the number of intervals for the entire set of data. Some statisticians suggest from 10 to 15 intervals and interval sizes of 2, 3, 5, 10, 20, etc. Using the raw data from Table II the range is 98 - 72 = 26 which means that only an interval size of 2 will give a number of intervals between 10 and 15.

Set up a frequency table for the data with a class interval width of 2 by using the following method.

Step 1: The first interval in descending order should include the largest data item. This gives us a choice of 98-99 or 97-98 as the first interval with 97.5 and 99.5 exact lower and exact higher limits respectively in the former case and 96.5 and 98.5 exact limits in the latter case.

Step 2: After choosing one of the options in 1, continue to write class intervals of width 2 until you reach an interval that contains 72, the smallest value.

Step 3: Record appropriate tally marks and their sums under the columns marked "Tally" and "F" for frequency in the table below.

TABLE III

Grouped Data

Class Interval	Tally	F
98 - 99		
96 - 97		

Check your answer with the completed one in the Appendix.

Frequency Polygon of Grouped Data

Supporters of the statement, "A picture is worth a thousand words," would prefer to see the data represented in the Frequency Table of Ungrouped or Grouped Data in pictorial form. To facilitate this effort, statisticians invented a frequency polygon to show the relationship between frequencies and class intervals.

Activity

Using the frequency table of the Grouped Data (Table III), execute the following:

1. Draw a horizontal line on a sheet of paper.

2. Select a scale which begins with the class interval immediately smaller than the interval containing the smallest data item and continue through the class interval immediately larger than the largest data item.

3. Determine the midpoints of each of the class intervals by adding each exact lower limit to the exact upper limit and dividing the sum by 2.

4. Draw a vertical axis such that its lowest endpoint intersects the horizontal axis at its left endpoint. Choose a vertical scale for frequencies so that 0 is the coordinate of the horizontal axis and the largest number of frequencies in the data has as its coordinate a point. Let this point's distance from the zero on the vertical scale be approximately three-fifths as long as the total width of the class intervals on the horizontal axis.

5. Plot the frequency point for each class interval on the vertical line which contains the midpoint of the class interval.

6. Connect the plotted points of adjacent intervals by line segments. You should note that the first and last intervals have frequencies of zero.

7. This line diagram is called a frequency polygon of grouped data.

8. Check your diagram with the frequency polygon given as Table IV in the Appendix.

Practice Problem

Construct a frequency polygon in grouped form with the following data:

{ 5, 7, 3, 22, 27, 34, 17, 19, 22, 17, 18, 9
10, 18, 21, 23, 30, 31, 11 }

Range, Variance, Standard Deviation

Frequency polygons give a graphic representation of a set of data, but they do not give one a quantitative idea of how much the data is scattered or dispersed. For this purpose there are three statistical measures: the range, variance, and standard deviation.

The range is the difference between the highest data item and the lowest data item. Compute the range for each set of data below. The first one is done for you.

(a) {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} range = 10 - 1 = 9

(b) {4, 4, 4, 4, 4, 4, 4, 4, 4, 4} range = ____ = ____

(c) {10, 12, 14, 16, 18, 20, 22, 24, 26, 28} range = ____ = ____

(d) {3, 8, 4, 2, 7, 9, 12, 5, 6, 4} range = ____ = ____

(e) {1, 2, 1, 2, 1, 2, 1, 3, 1, 2} range = ____ = ____

Since the range is determined by only two of the scores (no matter how long the list is), the range is not a good measure of how far apart the individual scores are or how much the scores vary within a given distribution. For this aspect of dispersion the variance and standard deviation (σ) are used. A relatively small standard deviation indicates that the scores tend to cluster about the mean but a larger standard deviation indicates that the values are more dispersed.

The steps for determining the variance of ungrouped data are:

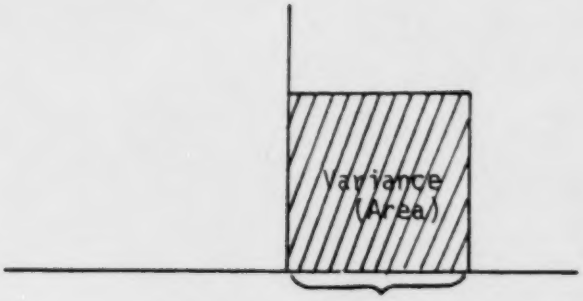
Step 1: Compute the mean ($\bar{x}$) of the data.

Step 2: Determine the deviation of each score from the mean.

Step 3: Square the deviations obtained in Step 2.

Step 4: Compute the mean of the squared deviations.

The standard deviation is the square root of the variance and is usually used because its geometric representation is linear, while the variance is represented by plane areas. (See Figure 2)



Standard Deviation (length only)

Example:

Compute the variance and the standard deviation for the following list of numbers of magazine subscriptions per household.

{ 0, 1, 2, 2, 3, 3, 3, 4, 6, 6 }

TABLE IV

Computation of the Variance

(1) Scores	(2) Deviations	(3) $(\text{Deviations})^2$	(4) Sum of D^2
0	$3 - 0 = 3$	$3^2 = 9$	9
1	$3 - 1 = 2$	$2^2 = 4$	4
2			
2			
3			
3			
3			
4	$3 - 4 = -1$	$(-1)^2 = 1$	1
6			
6			
10)	(		10) (

Refer to the steps below to complete Table IV.

Step 1: Compute the mean of the scores.

$$(30/10 = 3 = \bar{x})$$

Step 2: The deviations from the mean are found by subtracting each score from the mean. This is done in column (2) of the table. Notice the negative results when a score is larger than the mean. (complete column (2) at this point.)

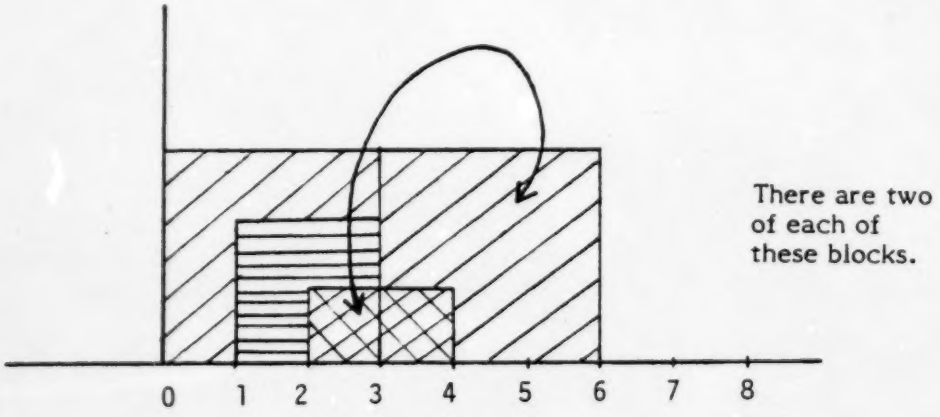
Step 3: To square the deviations, multiply each deviation by itself. This is done in column (3) of the table. (Complete column 3 at this point.)

Step 4: Divide the sum of the squared deviations by 10. After you have done this in column 4 of the table, your results should be 3.4 . (Complete column 4 now.)

By completing the work in step 4, you have found the variance. A completed table appears in Appendix A for you to check your work. Remember to ask your teacher or laboratory assistant for help if you need it.

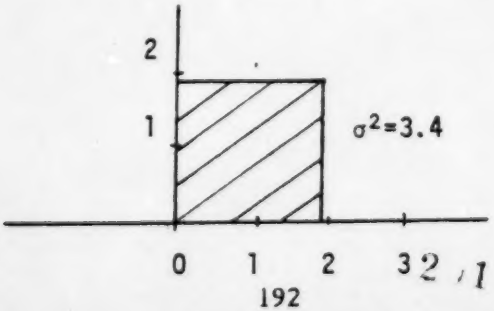
The standard deviation for the scores in Table IV is the square root of 3.4. Use a calculator to check that it is approximately 1.84 .

A geometrical representation of the preceeding results appears below in Figure 3 .



There are 3 squares with area of 0 since three scores equal the mean score of three. The variance is the sum of these areas divided by 10. Hence,

$$\sigma^2 = \{9 + 9 + 9 + 4 + 1 + 1 + 1 + 0 + 0 + 0\} \div 10 = 3.4$$



With a variance of 3.4 any one of the side lengths is the standard deviation. This length is computed as the square root of 3.4 and is approximately 1.84.

Exercise:

1. Compute the variance and standard deviation for

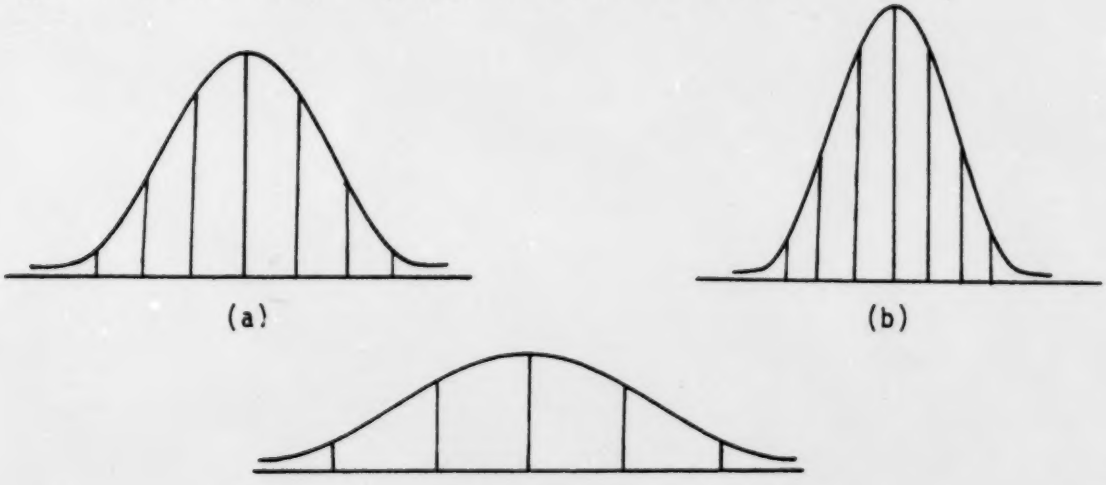
   0, 0, 1, 1, 2, 2, 6, 6, 6, 6, .

2. Compare the results in (1) with the results from your work in Table IV.

Normal Curve

The normal curve is a theoretical concept derived from a mathematical equation, quite apart from any concrete set of data. Its important feature accurately reflects the distribution of many sets of data, the values of which vary solely due to chance.

Figure 4 illustrates three representations of normal curves.



(a) (b)

(c)

The symmetry of these curves leads one to conclude that the mode, median, and mean are identical. If one considers the area bounded by the curve and the horizontal axis, it is quite evident that the middle line divides that area into halves, i.e., 50 percent of the area under the curve is left of the middle score (median, mean).

If we have a set of data whose frequency polygon represents a normal curve then we call the distribution a normal distribution or say the data is normally distributed. The distance between consecutive vertical lines in the representations above is called a standard deviation of the normal distribution.

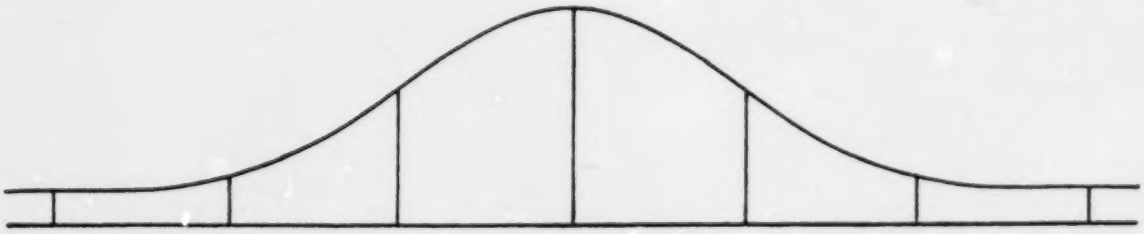
Characteristics of Normal Distributions

A. The area bounded by a standard deviation above the mean (middle line), a standard deviation below the mean, the normal curve, and the horizontal axis is 68.2 percent of the area under the curve.

B. The area bounded by two standard deviations above the mean, two standard deviations below the mean, the curve, and the horizontal axis is approximately 95 percent of the area under the curve.

C. The area bounded by three standard deviations above the mean, three standard deviations below the mean, the curve, and the horizontal axis is approximately 99 percent of the area under the curve.

D. Percentile ranks are related to mean and standard deviations as given in Figure 5 below.



	$x-3\sigma$	$x-2\sigma$	$x-\sigma$	x	$x+\sigma$	$x+2\sigma$	$x+3\sigma$
Standard scores							
PERCENTILE	.1	2.3	15.9	50	84.1	97.7	99.9

Example:

Suppose we are given a set of normally distributed data with mean 70 and standard deviation 6 and asked to locate the percentile rank for a score of 82. We first determine the number of standard deviations the score is from the mean. In this case 82 is 12 above the mean and 12 divided by 6 is 2, so we have a score which is two standard deviations above the mean. Figure 5 shows that such a score is at the 97.7 percentile for the data. If we had selected a score like 79 we would have found it to be 9 units above the mean and with standard score of 9 divided by 6 or, 1.5. Our figure does not accurately reflect the percentile rank of $x + 1.5$, but we can say that it is between the 84.1 and 97.7 percentile ranks. Suppose the score had been 64. It would then be 6 units smaller than the mean or one standard deviation below the mean; Figure 5 shows its percentile rank is 15.9. Most statistics books have a table of "z" scores for the unit normal distribution ($\bar{x} = 0$ and $\sigma = 1$) with which one can find the percentile rank of any deviation score.

Another interesting characteristic of the normal curve is that it enables one to find percentages of scores between deviation scores. Figure 5 shows that a score which is one deviation above the mean is at the 50th percentile. Hence, 34.1 percent of the scores are between the mean and one deviation above it.

Exercises:

Answer the following for a Normal Distribution.

1. What percent of the scores are between one deviation below the mean and 2 deviations above the mean? _____

2. What percent of the scores are greater than one deviation above the mean? _____

3. If the mean is 80 and the standard deviation is 7, what percent of the scores are between 73 and 94? _____

4. If the mean is 65 and the standard deviation is 6, what is the percentile rank of 47? _____

Post-Test A

A pair of dice was rolled ten times. Use the results from the rolls to answer questions 1 through 6.

{6, 6, 7, 4, 12, 11, 6, 7, 10, 11}

1. The mean for the set of data is

   (a) 8 (b) 11.5 (c) 6 (d) 7.5

2. The median for the set of data is

   (a) 6 (b) 11.5 (c) 8 (d) 7

3. The mode for the set of data is

   (a) 12 (b) 7 (c) 6 (d) 11

4. The midrange for the ten rolls is

   (a) 7.5 (b) 8 (c) 11.5 (d) 6

5. The standard deviation to the nearest tenth is

   (a) 6.8 (b) 2.6 (c) 68 (d) 8.2

6. The variance to two decimal places is

   (a) 6.86 (b) 6.80 (c) 6.76 (d) 6.79

Use the following set of 20 heights (in inches) to answer questions 7 through 10.

{66, 63, 36, 38, 68, 68, 65, 64, 64, 71, 75,

51, 67, 68, 46, 67, 36, 65, 66}

7. Construct a frequency table of the data.

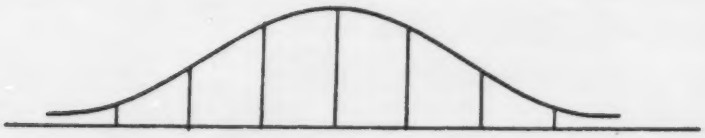
8. Construct a frequency table of the data grouped in intervals of length 3.

9. Construct a frequency polygon of the data as grouped in problem 8.

10. The percentile rank of a raw score of 68 is

    (a) 14.3 (b) 65 (c) 75 (d) 70

Use Figure 6 for problems 11 and 12.



	$x-3\sigma$	$x-2\sigma$	$x-\sigma$	x	$x+\sigma$	$x+2\sigma$	$x+3\sigma$
PERCENTILE	.1	2.3	15.9		84.1	97.7	99.9

11. Find the percentile rank for a score of 71 from a normally distributed set of data with mean 62 and standard deviation 6.

(a) 84.1 (b) 97.7 (c) 99.9 (d) 93.3

12. Find the percentile rank for a score of 48 from a normally distributed set of data with mean 56 and standard deviation 4.

(a) 97.7 (b) 2.3 (c) 15.9 (d) 10.2

Post-Test B

An experiment consists of flipping 10 coins and recording the number of heads. The results as listed below should be used to answer questions 1 through 6.

{3, 6, 2, 5, 7, 7, 8, 4, 1}

1. The mean number of heads is

   (a) 7 (b) 4.5 (c) 5 (d) 5.5

2. The median number of heads is

   (a) 5 (b) 5.5 (c) 7 (d) 4.5

3. The mode for the data is

   (a) 4.5 (b) 5.5 (c) 7 (d) 8

4. The midrange for the data is

   (a) 4.5 (b) 5 (c) 7 (d) 5.5

5. The variance to two decimal places is

   (a) 5.20 (b) 2.28 (c) 7 (d) 52.00

6. The standard deviation to the nearest tenth is

   (a) 2.2 (b) 5.3 (c) 5.2 (d) 2.3

Use the following set of test data to answer questions 7 through 10.

{25, 25, 30, 35, 40, 45, 45, 45, 50, 50, 60,
65, 65, 70, 70, 70, 80, 90, 95 }

7. Construct a frequency table of the data.

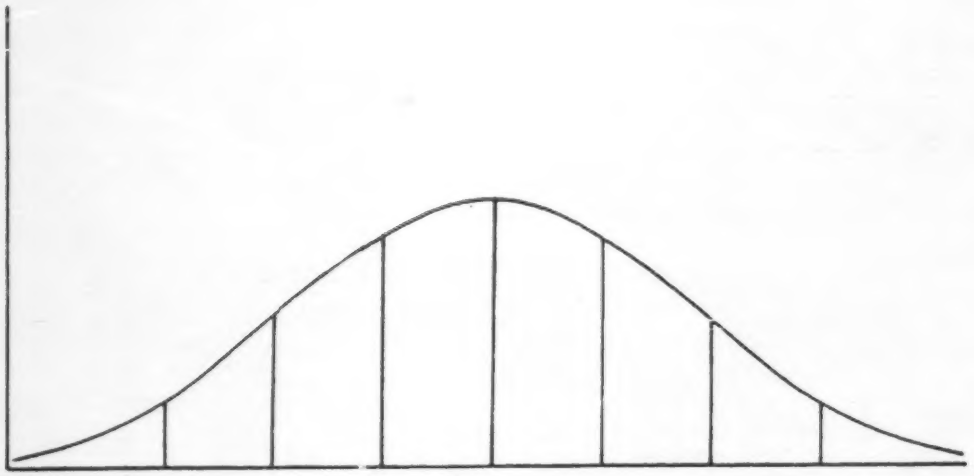
8. Construct a frequency table of the data grouped in intervals of length 5.

9. Construct a frequency polygon of the data as grouped in problem 8.

10. Find the percentile rank of a raw score of 65.

    (a) 40 (b) 60 (c) 70 (d) 12

Use Figure 7 for problems 11 and 12.



	$\bar{x}-3\sigma$	$\bar{x}-2\sigma$	$\bar{x}-\sigma$	$\bar{x}$	$\bar{x}+\sigma$	$\bar{x}+2\sigma$	$\bar{x}+3\sigma$
Percentile	.1	2.3	15.9	50	84.1	97.7	99.9

11. Find the percentile rank for a score of 400 from a normally distributed set of data with mean 500 and standard deviation 100.

(a) 2.3 (b) 15.9 (c) 84.1 (d) 34.1

12. Find the percentile rank for a score of 85 from a normally distributed set of data with mean 65 and standard deviation 5.

(a) 97.7 (b) 98.5 (c) 99.9 (d) .1

References

Cahen, Leonard and Hansen, Duncan. Applications and Problems in Elementary Statistics. Brooks/Cole: Belmont, California, 1963.

Elzey, Freeman. Introduction to Statistics. Brooks/Cole: Belmont, California, 1971.

McCollough, Celeste and Atta Loche. Statistical Concepts: A Program for Self-Instruction. McGraw-Hill: New York, 1963.

Sanders, Donald, Murph, A.F., and Eng, R.J., Statistics: A Fresh Approach. McGraw-Hill: New York, 1976.

APPENDIX:

BASIC STATISTICS

TABLE II - COMPLETED VERSION

Mathematics Placement Test Scores (%ile)
(Frequency Table)

Score	Tally	Frequency	Score	Tally	Frequency
70		0	85	||	2
71		0	86		0
72	|||	3	87	|	1
73	||	2	88	|	1
74		0	89		0
75	|	1	90	||||	4
76		0	91	|||	3
77	|	1	92	||	2
78	||||	4	93	|||	3
79	|	1	94	|	2
80	||	2	95	|||	3
81	~~||||~~	5	96	|	1
82	~~||||~~	5	97	|	1
83	|	1	98	|	1
84	|	1	99		0
					N = 50

TABLE III - COMPLETED VERSION

Grouped Data

Class Interval	Tally	F
98-99	|	1
96-97	||	2
94-95	𝍸	5
92-93	𝍸	5
90-91	𝍸 ||	7
88-89	|	1
86-87	|	1
84-85	|||	3
82-83	𝍸 |	6
80-81	𝍸 |	7
78-79	𝍸	5
76-77	|	1
74-75	|	1
72-73	𝍸	5
		N = 50

TABLE IV - COMPLETED VERSION

Computation of the Variance

(1) Scores	(2) Deviations	(3) $(\text{Deviations})^2$
0	$3 - 0 = 3$	$3^2 = 9$
1	$3 - 1 = 2$	$2^2 = 4$
2	$3 - 2 = 1$	$1^2 = 1$
2	$3 - 2 = 1$	$1^2 = 1$
3	$3 - 3 = 0$	$0^2 = 0$
3	$3 - 3 = 0$	$0^2 = 0$
3	$3 - 3 = 0$	$0^2 = 0$
4	$3 - 4 = 1$	$(-1)^2 = 1$
6	$3 - 6 = 3$	$(-3)^2 = 9$
6	$3 - 6 = 3$	$(-3)^2 = 9$
$10\overline{)\ 30\ }3$		

(4) Sum of D^2

9

4

1

1

0

0

0

1

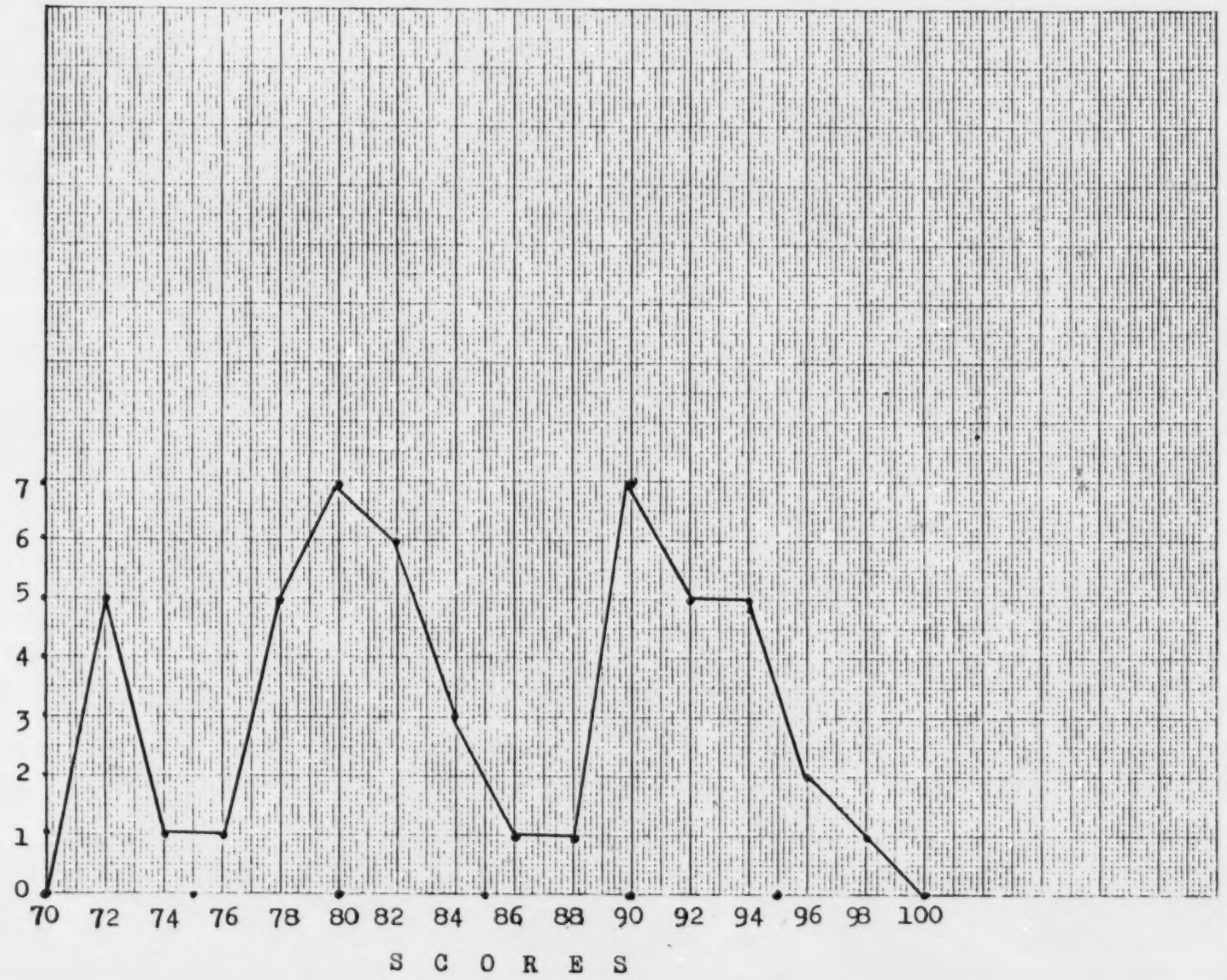
9

9

$10\overline{)\ 34\ }3.4$

TABLE V FREQUENCY POLYGON OF GROUPED DATA

No. 1520 Metric Graph Paper



ELEMENTS OF NUMERICAL TRIGONOMETRY

Introduction

This section introduces you to the study of trigonometry, i.e., the study of measurement in triangles. It will give you the opportunity to apply many of the skills that you have developed in prior sections and, at the same time, to take a look at a fascinating subject in itself.

Objectives

When you have completed this section you should be able to do the following:

(1) Identify by name the six fundamental trigonometric ratios.

(2) Use the graphical method to approximate the trigonometric ratios for any given positive acute angle with an accuracy of two decimal places.

(3) Given the coordinates of a point in the first quadrant, use the graphical method to compute the trigonometric ratios for the acute angle determined by the given point and the origin.

(4) Given a right triangle, the length of one if its sides and the measure of one of its angles, find the lengths of the remaining sides by selecting appropriate trigonometric ratios.

(5) Given a right triangle and the lengths of two of its sides, find the length of the third side and the measures of all three of its angles.

(6) Solve any simple word problem involving a right triangle if (a) the measure of one angle and the length of one side are given; or (b) the lengths two or more sides are given.

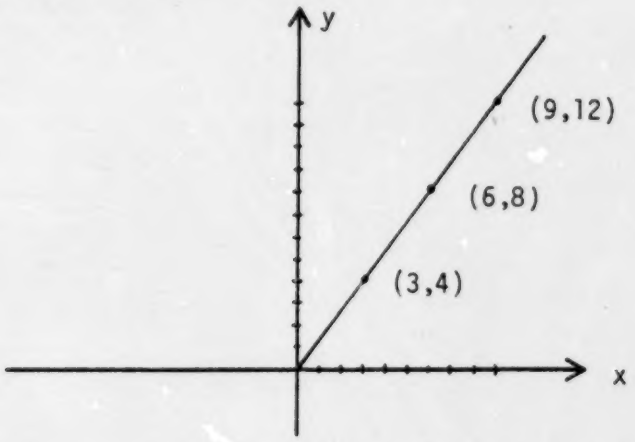
Study Material

Trigonometric Ratios. Let's begin by considering the fraction $\frac{3}{4}$.

What are some positive fractions equivalent to $\frac{3}{4}$?

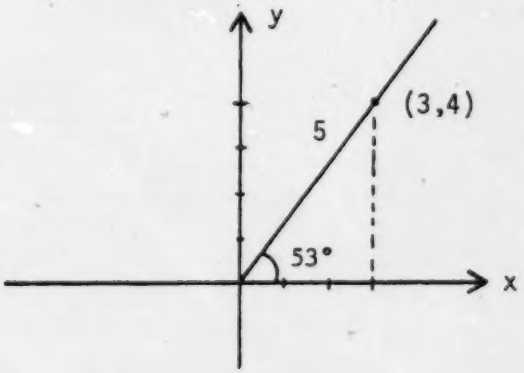
$$\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{16} = \cdots$$

For each of these fractions take the numerator and the denominator to be coordinates of points: (3,4), (6,5), (9,12) .... and make an extended graph of these points. What kind of graph do you have? Answer: a straight line passing through the origin.



From this you can see that the fraction $\frac{3}{4}$ and the point (3,4) determine an angle with the positive x axis formed by connecting (3,4) with the origin. (Note that the terminal side of the angle contains points associated with fractions equivalent to $\frac{3}{4}$.) What is the measure of this angle? Use a protractor to verify that the angle is approximately 53°. Let's call it 53° for the purpose of this discussion.

Now, by dropping a perpendicular from the point (3,4) to the x axis, you see that (3,4) actually determines a right triangle (see figure below). What are the lengths of the sides of this triangle? Use the Pythagorean theorem to show that the lengths are: 3, 4, 5.

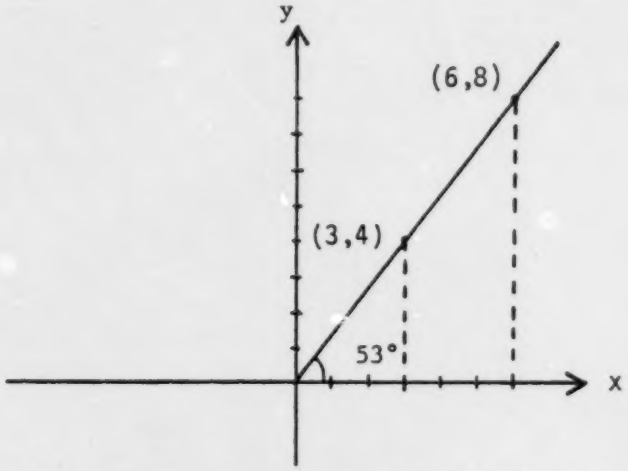


How many different ratios can you form from the numbers, 3, 4, and 5, taking them two at a time? Answer: six. They are:

$$\frac{4}{5}, \quad \frac{3}{5}, \quad \frac{4}{3}, \quad \frac{5}{4}, \quad \frac{5}{3}, \quad \frac{3}{4}$$

These six ratios are very closely related to the angle 53°, for if you change the ratios, the angle will change, becoming larger or smaller depending on how the lengths of the sides of the triangle are altered.

Next select a different point on the terminal side of the angle, say (6,8) and draw the triangle determined by (6,8) as in the figure below:



Using the Pythagorean theorem again you can show that the hypotenuse of the larger triangle is 10, and that the ratios for this triangle are:

$$\frac{8}{10}, \quad \frac{6}{10}, \quad \frac{8}{6}, \quad \frac{10}{8}, \quad \frac{10}{6}, \quad \frac{6}{8}$$

But by reducing fractions, you see that these are the same ratios we found before:

$$\frac{4}{5}, \frac{3}{5}, \quad \frac{4}{6}, \quad \frac{5}{4}, \quad \frac{5}{3}, \quad \frac{3}{4}$$

In fact, it doesn't matter which point you choose on the terminal side of the angle 53°, you will always get the same six ratios. Try it for some different points and see. Why is this true? (Hint: the triangles that you have been drawing are similar triangles. Explain.)

Thus, the angle 53° is determined by these six ratios and vice versa; because of this we give these ratios the following special names:

$$\text{sine } 53^\circ = \frac{4}{5} \qquad \text{cosecant } 53^\circ = \frac{5}{4}$$

$$\text{cosine } 53^\circ = \frac{3}{5} \qquad \text{secant } 53^\circ = \frac{5}{3}$$

$$\text{tangent } 53^\circ = \frac{4}{3} \qquad \text{cotangent } 53^\circ = \frac{3}{4}$$

and we refer to them as trigonometric ratios or trigonometric functions for the angle 53°.

Exercise: Use the preceding graphical method to find the trigonometric ratios for the angle determined by the point (5,12), and find the measure of the angle.

Answer: The angle is approximately 67° and the ratios are:

$$\text{sine } 67^\circ = \frac{12}{13} \qquad \text{cosecant } 67^\circ = \frac{13}{12}$$

$$\text{cosine } 67^\circ = \frac{5}{13} \qquad \text{secant } 67^\circ = \frac{13}{5}$$

$$\text{tangent } 67^\circ = \frac{12}{5} \qquad \text{cotangent } 67^\circ = \frac{5}{12}$$

Let us now reverse our process somewhat. Begin by drawing the angle 70° on a piece of graph paper. Next, approximate the coordinates of a point on the terminal side of the angle, and drop a perpendicular from this point, forming a right triangle as before. What are the values of the trigonometric ratios for 70°? Three of the answers are approximately:

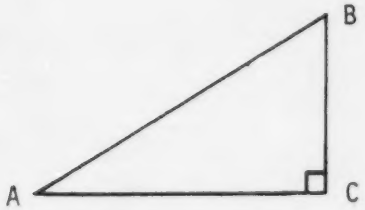
$$\sin 70^\circ = .93$$

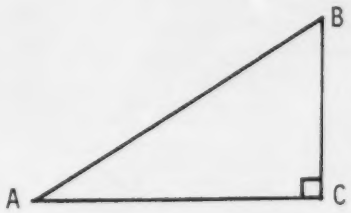
$$\cos 70^\circ = .34$$

$$\tan 70^\circ = 2.74$$

Here we used standard abbreviations for the ratio names. Can you figure out how to get the remaining three ratios from these? (Check the previous examples for a pattern.) We will use only the sine, cosine and tangent in the remainder of this section.

General Form for The Trigonometric Ratios. The foregoing discussions suggest the following general forms for the trigonometric ratios of any positive acute angles of measures θ or β.





$$\sin\theta = \frac{\text{side opposite } \theta}{\text{hypotenuse}} = \frac{BC}{AB} \qquad \sin\beta = \frac{\text{side opposite } \beta}{\text{hypotenuse}} = \frac{AC}{AB}$$

$$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{AC}{AB} \qquad \cos\beta = \frac{\text{side adjacent } \beta}{\text{hypotenuse}} = \frac{BC}{AB}$$

$$\tan\theta = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta} = \frac{BC}{AC} \qquad \tan\beta = \frac{\text{side opposite } \beta}{\text{side adjacent to } \beta} = \frac{AC}{BC}$$

A **Trigonometric Table.** Using a finely grided piece of graph paper (e.g, J.L. Hammet Company, No 1520 Metric Graph Paper) find the trigonometric ratios for the angles, 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°, as follows:

1. Use a compass to draw a large semicircle at the base of the paper (use a radius of 100 small units on the Hammet No. 1520).

2. Use a protractor to construct the angles using the center of the semicircle as the origin.

3. For each angle, drop a perpendicular from the point where the terminal side intersects the semicircle, and form a right triangle as in the previous examples. (Note that the hypotenuse of the right triangle is 100 units long. Why?)

4. Compute the ratios for each angle and make a table of the results; round off your answers to the nearest hundredth.

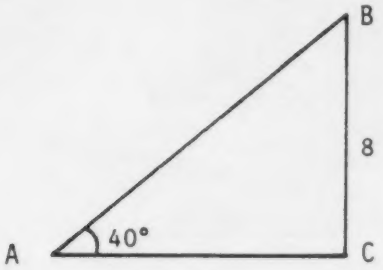
5. Compare your results with the chart below.

Trigonometry Table

Angle	Sine	Cosine	Tangent
10°	.17	.98	.19
20°	.94	.94	.36
30°	.50	.86	.57
40°	.64	.75	.83
50°	.76	.64	1.19
60°	.86	.50	1.75
70°	.94	.34	2.74
80°	.98	.17	5.67

If you examine the trigonometric tables in the rear of any trigonometry textbook you will find the trigonometric functions for many more angles with much greater accuracy. But you should now have some idea of how such tables are made.

Use of the Trigonometry Table in Problem Solving. Solve the right triangle ABC below (i.e., find all of its missing parts).



Since $A + B = 90^{\circ}$, we have $B = 90^{\circ} - A = 90^{\circ} - 40^{\circ} = 50^{\circ}$ (Recall that the sum of the angles in any triangle equals 180°). To find the lengths of AB and AC we select appropriate trigonometric functions, each involving only one unknown quantity. For AB we use:

$$\text{sine } 40^{\circ} = \frac{BC}{AB} = \frac{8}{AB}$$

From the trigonometric table, we know that $\sin 40^{\circ} = .64$, therefore

$$.64 = \frac{8}{AB}$$

$$.64AB = 8$$

$$AB = \frac{8}{.64} = 12.5$$

For AC we use:

$$\tan 40^{\circ} = \frac{BC}{AC} = \frac{8}{AC}$$

Using the table again we find that $\tan 40^{\circ} = .83$, therefore

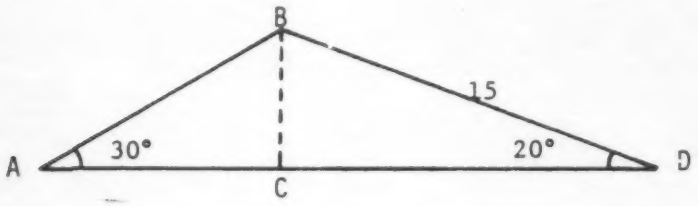
$$.83 = \frac{8}{AC}$$

$$.83\ AC = 8$$

$$AC = \frac{8}{.83} = 9.64$$

(Note: We could have used $\cos 40^\circ = \frac{AC}{AB}$ to find the length of AC.)

Exercise: Find the missing parts in triangle ABD below:



Solution: $\text{sine } 20^\circ = \frac{BC}{15}$

$BC = 15 \sin 20^\circ = 15\,(.34) = 5.10$

Now use the following trigonometric functions to find the remaining sides:

$$\cos 20^\circ = \frac{CD}{15}$$

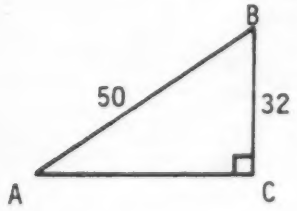
$$\sin 30^\circ = \frac{BC}{AC}$$

$$\tan 30^\circ = \frac{BC}{AC}$$

The answers are approximately: CD = 14.10, AB = 10.20, AC = 8.95,

angle ABC = 60°, angle DBC = 70°

Exercise: Find the missing angles in triangle ABC below:



Even though angles A and B are unknown, we can proceed by selecting the appropriate trigonometric function:

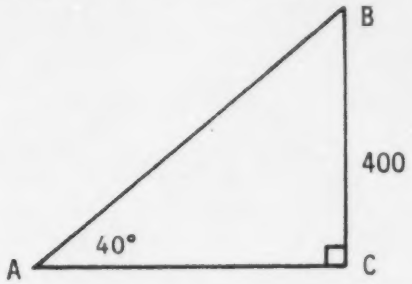
$$\sin A = \frac{32}{50} = .64$$

Now which angle in the table has a sine equal to .64? Answer: 40°. Therefore, $A = 40^\circ$. This says that $B = 90^\circ - A = 50^\circ$. The remaining parts can now be found as before.

Word Problems

Example: An airplane takes off and climbs at an angle of 40° to the ground. How far is the airplane from the airport when it reaches an altitude of 400 feet?

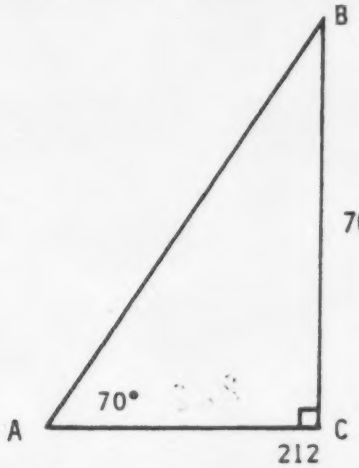
Solution: The problem can be diagrammed as follows:



to find the length of AB use $\sin 40^\circ = \frac{400}{AB}$.

Answer: AB = 625 ft.

Example: A man is standing atop a 100-foot building. If a woman on the ground must incline her head at an angle of 70° in order to see the man, how far is she standing from the building?

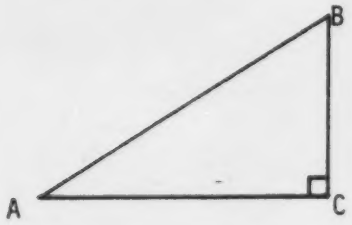


Hint: Use $\tan 70^\circ = \frac{700}{AC}$

Answer: AC = 255.47 ft. (approximate)

Exercise. Find the missing parts in problems (1) - (8) below.

Problems (1) - (5) refer to triangles ABC of the form:



(1) A - 40°, AB = 20

(2) B = 20°, AC = 18

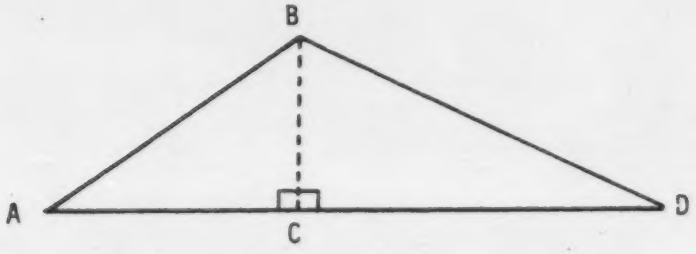
(3) B = 70°, AB = 12

(4) AB = 200, AC = 126

(5) AC = 30, BC = 16

(6) A = 60°, BC = 18

Problems (6) and (7) refer to triangles of the form:



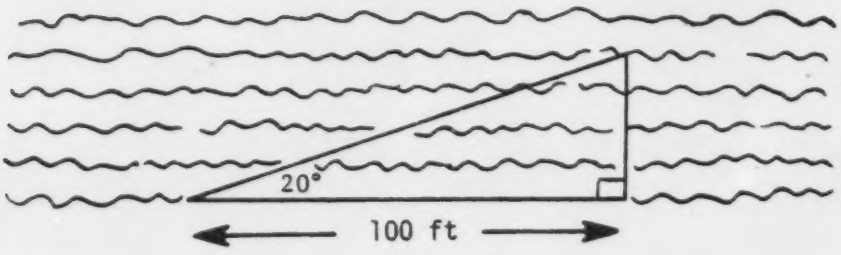
(6) $A = 30^{\circ}$, $D = 30^{\circ}$, $BC = 8$

(7) $A = 30^{\circ}$, $D = 40^{\circ}$, $BC = 8$

(8) $A = 40^{\circ}$, $D = 50^{\circ}$, $AC = 10$

(9) A 100-foot ladder is leaning against a wall, and it touches the wall at a point 34 feet from the ground. What angle does the ladder make with the ground?

(10) Find the width of the river in the diagram below:



Answers

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

(9)

(10)

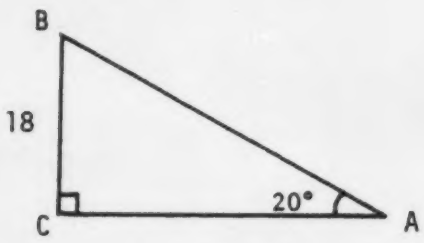
Post-Test

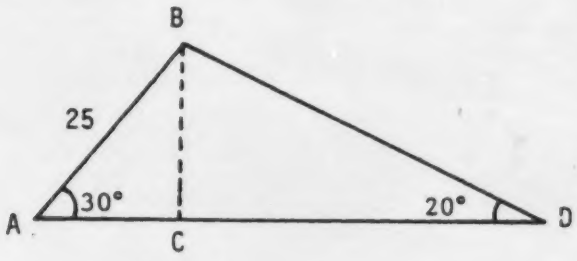
1. Name the six trigonometric ratios which can be found for any positive acute angle.

2. Given the point (7,24), graph the point and draw the angle determined by this point and the origin. Find the measure of the angle with a protractor and write the six trigonometric ratios for this angle.

3. Approximate, using the graphical method, the trigonometric ratios for the angles 30°, 47°, 76°.

4. Find the missing parts in the triangles below:





5. At a point 250 feet from the base of a building, the angle of elevation to the top of the building is 50°. How high is the building?